BUOYANT PLUME RISE CALCULATED BY LAGRANGIAN AND EULERIAN MODELLING

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INTRODUCTION

Plumes emitted from stacks are usually buoyant due to temperature differences to the ambient air. Such buoyancy effects may influence strongly the turbulent mixing of the plume with the ambient air. This is essential for reactive plumes, because the mixing of compounds distributed in the ambient air and the plume determines the occurrence of reactions. Furthermore, changes in the turbulent mixing of plume and ambient flow may affect considerably the spatial patterns of the distribution of tracers and consequently their ground concentration.

These mixing processes can be described in the Eulerian approach, i.e. based on the conservation equations of mass, momentum and thermal energy, by means of assumptions on entrainment and extrainment. The idea of entrainment of air into plumes by plume-generated turbulence permits the explanation of the "two-thirds" power law for the buoyant plume rise, which is observed in a neutral stratified flow without significant turbulence.[1] For a turbulent atmosphere one finds a levelling off of the plume, that means the plume follows in the initial stage the "two-thirds" power law and at later times its mean height becomes constant. This transition and the final plume height can be explained by an extrainment proposed by Netterville, i.e. entrainment of plume material into the surrounding fluid due to the ambient turbulence.[2] This approach is able to describe the plume behaviour in complex flows, that means for an arbitrary stratification and different intensities of ambient turbulence. The applicability of this model was proved by means of field observations of plume trajectories and final rise and it was found to perform well.[2,3]

But there are two problems related to this approach: Firstly, in particular the extrainment concept requires the knowledge of parameters which cannot be derived directly from measurements, i.e. they have to be estimated with different ad hoc assumptions which are not easy to justify.[3] Also large-eddy simulations are not instructive for the explanation of these mixing processes. Turbulence processes within the plume cannot be correctly simulated, because they occur on a subgrid scale.[4,5] Secondly, the dispersion problem is not solved in this way. The plume is calculated as a distributed line source and a dispersion model is needed to calculate the spreading of material of this line source. This is a non-trivial problem, because of the uncertainty of this 'source strength'.

The Lagrangian approach to this problem is able to simulate both the mean plume behaviour and the dispersion of plume material as observed by van Dop.[6] The problem to explain the turbulent mixing of the plume and the ambient fluid appears here as the problem to estimate the time behaviour of the time scales which determine the dynamics of particle motion. An approach to solve this question consists in the explanation of buoyant turbulence as stochastic motion of particles and change of their temperatures.[7] Accordingly, the construction of particle models in consistency with budget equations of the turbulence permits firstly the explanation of the observed plume features, secondly, the explanation of parameters in the Eulerian approach which cannot be derived directly from measurements, and thirdly, the description of the dispersion of tracers.

These features of the Lagrangian approach are considered here in comparison with the Eulerian approach. This is done by explaining, how buoyant turbulence can be represented by a stochastic particle model. Then, the mean plume rise is calculated by these Lagrangian equations and these findings are compared with the results of the Eulerian approach.

BUOYANT TURBULENCE AS STOCHASTIC PARTICLE MOTION

One approach to explain buoyant turbulence as stochastic particle motion consists in the estimation of Lagrangian solutions of turbulence budget equations up to second-order.[7] Here, the coefficients of linear Lagrangian equations for the motion and temperatures of particles are chosen, such that budget equations of turbulence are fulfilled for the means and all the variances of the coupled Eulerian velocity-temperature fields. Consequently, the particle move and change their temperatures, where these turbulence budgets are satisfied at any time and at arbitrary positions in the flow.

Let us consider for simplicity only the vertical motion of particles in an Eulerian flow field with a mean horizontal velocity U into the x^1-direction and a mean potential temperature Θ, which depend only upon the vertical coordinate x^3. The mean vertical velocity is neglected. Lagrangian equations for the change of the particle height $x_L{}^3$ over the source, the vertical velocity $U_L{}^3$ and the potential temperature Θ_L in time t (L denotes a Lagrangian quantity) read then according to the above described approach

$$\frac{d}{dt}\langle x_L{}^3\rangle = \langle U_L{}^3\rangle, \tag{1a}$$

$$\frac{d}{dt}\langle U_L{}^3\rangle = -\frac{k_1}{4\tau}\langle U_L{}^3\rangle + \beta g\langle(\Theta_L - \Theta)\rangle, \tag{1b}$$

$$\frac{d}{dt}\beta g\langle(\Theta_L - \Theta)\rangle = -\frac{2k_3 - k_1}{4\tau}\beta g\langle(\Theta_L - \Theta)\rangle, \tag{1c}$$

where <···> denotes an ensemble average. The Eulerian quantity Θ is estimated at fixed heights x^3, which are replaced by the actual particle $x_L{}^3$ in the above equations. Furthermore, β is the thermal expansion coefficient, g is the acceleration due to gravity and k_1, k_3 and k_4 are parameters arising from the applied closure assumptions.[7] The values k_1 = 8.3, k_3 = 6.5 and k_4 = 4.0 are applied for these parameters in the calculation below. The essential quantity to be estimated is the dissipation time scale of turbulence $\tau = q^2 / (2\langle\varepsilon\rangle)$, where q^2 is twice the turbulent kinetic energy (TKE) and $\langle\varepsilon\rangle$ denotes the mean dissipation rate of TKE.

These equations have the same structure as those derived by van Dop,[6] but here the coefficients are estimated by the consistency with the variance budget equations. In particular the estimation of the time scales for the vertical motion of particles and their temperature changes is essential, because the time behaviour of these quantities determines the plume rise features. Usually, the particles start with a much smaller time scale than that of the ambient turbulence. This initial stage of buoyant plume rise is characterized under neutral stratification and a calm ambient turbulence by the "two-thirds" power law, that means the mean plume height grows proportional to $t^{2/3}$. At later times, τ approaches gradually to the value of the time scale of the ambient turbulence, which causes a levelling off of the mean plume height. This characteristic plume behaviour was explained by van Dop by adopting ad hoc assumptions on the time scale behaviour in these two stages. This observation enables the simulation of particle motion in accord with the similarity behaviour, but this approach raises questions on the range of applicability of these relations, the estimation of required parameters and e.g. the reflection of stratification effects.

LAGRANGIAN PLUME RISE EQUATIONS

The above described explanation of buoyant turbulence by particle motion offers the possibility to calculate τ by means of concepts of turbulence theory. In order to demonstrate the main features of this approach let us assume, that the vertical shear $\partial U / \partial x^3$ of the mean horizontal wind is constant. It is now advantageous to consider combinations of t and τ with the vertical shear, i.e. the dimensionless quantities $t' = t \cdot \partial U / \partial x^3$ and $T = \tau \cdot \partial U / \partial x^3$ are introduced. By adopting standard methods for the estimation of the mean flow frequency τ^{-1} one obtains then[8]

$$\frac{d}{dt'}T = (C_{\varepsilon 2} - 1) - 2T \cdot (C_{\varepsilon 1} - 1) \cdot \left\{ -\frac{\hat{V}^{13}}{\hat{q}^2} + \frac{\hat{V}^{34}}{\hat{q}^2} \right\}, \tag{2}$$

where $C_{\varepsilon 1} = 1.56$ and $C_{\varepsilon 2} = 1.9$ are constants. $\hat{V}^{13}$, $\hat{V}^{34}$ and $\hat{q}^2$ are dimensionless variances which satisfy the equation system

$$\frac{d}{dt'}\begin{pmatrix} \hat{V}^{13} \\ \hat{V}^{14} \\ \hat{V}^{34} \\ \hat{V}^{33} \\ \hat{V}^{44} \\ \hat{q}^2 \end{pmatrix} = \frac{1}{T} \cdot \begin{pmatrix} -k_1/2 & T & 0 & -T & 0 & 0 \\ -RiT & -k_3/2 & -T & 0 & 0 & 0 \\ 0 & 0 & -k_3/2 & -RiT & T & 0 \\ 0 & 0 & 2T & -k_1/2 & 0 & (k_1-2)/6 \\ 0 & 0 & -2RiT & 0 & -k_4 & 0 \\ -2T & 0 & 2T & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \hat{V}^{13} \\ \hat{V}^{14} \\ \hat{V}^{34} \\ \hat{V}^{33} \\ \hat{V}^{44} \\ \hat{q}^2 \end{pmatrix}, \tag{3}$$

where the gradient Richardson number $Ri = [\beta g\, \partial\Theta / \partial x^3] / [\partial U / \partial x^3]^2$ appears, i.e. T can be calculated in this way also for unstably and stably stratified flow. The equation system (3) arises from the second-order moment equations which are proposed for the estimation of the Lagrangian equations (1a-c). The variances are normalized to twice the TKE at the initial time $q^2(t = 0)$, and the variances related to temperature fluctuations (indicated by the superscript 4) appear multiplied with a factor $\beta g\, (\partial U / \partial x^3)^{-1}$.

The first term on the right-hand side of (2) corresponds with the entrainment idea. T grows proportional to t', which is related to a power law for the buoyant plume rise.[6] It is worth emphasizing, that only this term appears in the original frequency equation of Kolmogorov.[8] The second term on the right-hand side of (2) depends on the state of turbulence. This term causes a decrease of T related to a levelling off of the plume (see below). These effects are just the result of the extrainment idea.

Introducing now the normalized particle height $Z = \langle x_L^3 \rangle (\partial U / \partial x^3)^2 / B_0$, the particle velocity $W = \langle U_L^3 \rangle (\partial U / \partial x^3) / B_0$ and the buoyancy $B = \beta g \langle (\Theta_L - \Theta) \rangle / B_0$, where $B_0 = \beta g \langle (\Theta_L - \Theta) \rangle (t = 0)$ is written for the initial buoyancy, the Lagrangian equations (1a-c) can be rewritten for a neutral stratification into

$$\frac{dZ}{dt'} = W, \tag{4a}$$

$$\frac{dW}{dt'} = -\frac{k_1}{4T} W + B, \tag{4b}$$

$$\frac{dB}{dt'} = -\frac{2k_3 - k_1}{4T} B. \tag{4c}$$

In order to investigate the conditions for the reproduction of the "two-thirds" power law let us consider the mean particle height Z over the source, which follows only from the first term on the right-hand side of (2). Z is then determined by

$$Z = \frac{I^{2-m_1}}{(C_{\varepsilon 2} - 1)^2 m_3} \cdot \left\{ \frac{\left(I + (C_{\varepsilon 2} - 1)t'\right)^{m_1} - I^{m_1}}{m_1} - \frac{\left(I + (C_{\varepsilon 2} - 1)t'\right)^{m_2} - I^{m_2}}{m_2} \cdot I^{m_3} \right\}, \tag{5}$$

where the abbreviations $m_1 = 2 - (2k_3 - k_1) / (4 [C_{\varepsilon 2} - 1)])$, $m_2 = 1 - k_1 / (4 [C_{\varepsilon 2} - 1)])$ and $m_3 = 1 - (k_3 - k_1) / (2 [C_{\varepsilon 2} - 1)])$ are applied and I is written for the initial value of T. For large times one obtains

$$Z = \frac{1}{m_3 m_1} \cdot \left(\frac{I}{C_{\varepsilon 2} - 1} \right)^{2-m_1} \cdot t'^{m_1}, \tag{6}$$

when I is neglected with respect to t' and the highest power of t' is only taken into account. Hence, the "two-thirds" power law is obtained, if $m_1 = 2/3$, i.e., if

$$k_3 = \frac{k_1}{2} + \frac{8}{3} \cdot (C_{\varepsilon 2} - 1). \tag{7}$$

Provided this condition, the similarity behaviour of the buoyant plume rise appears as a consequence of the explanation of the buoyant turbulence by stochastic particle motion and the original frequency equation of Kolmogorov.

The curve (5), which follows only from the initial time scale, is shown in Figure 1 together with the "two-thirds" power law curve (6) and the curve which follows from the solution of (4a-c) combined with (2) and (3). The latter one curve is indicated by HFM to point out to the use of the homogeneous (there are no spatial transport terms) frequency model (2) and (3). The equations (4a-c) are solved for the initial conditions $Z(t = 0) = 0$, $W(t = 0) = 0$ and $B(t = 0) = 1$ and the equations (3) are solved by a Runge-Kutta procedure and with initial conditions $\hat{V}^{ij} = 1/3\, \delta_{ij}$. The initial value $I = \tau_0\, \partial U / \partial x^3$ is set to be 0.173 in accord with data of the Nanticoke plume rise measurements.[2] The curve obtained by (5) coincides in the initial stage with the HFM-curve and approaches later to the "two-thirds" power law curve (6). The HFM-curve levels off due to ambient turbulence effects. This final plume height will be discussed below after considering corresponding relations of the Eulerian approach.

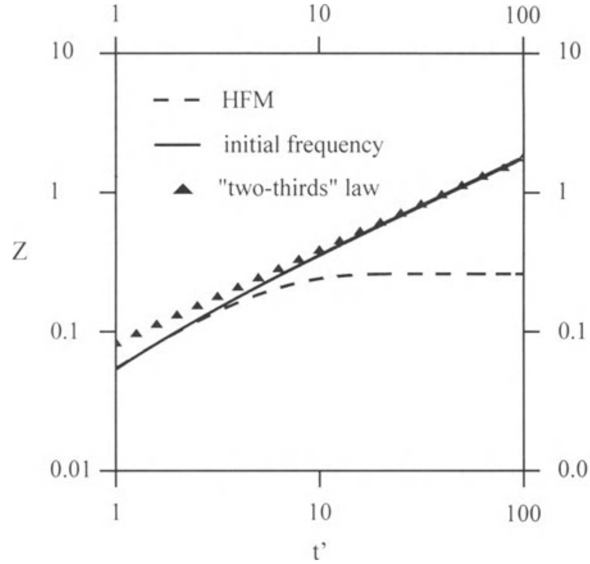

Figure 1. The dashed line gives the normalized height Z as function on t' as obtained by the equation (2). The solid curve represents the initial rise, which follows only by the initial particle frequency. The triangles represent the observed "two-thirds" power law.

EULERIAN PLUME RISE EQUATIONS

These obtained features will be compared now with the Eulerian plume rise model derived by Netterville.[2] Within this approach, the plume is considered as a superposition of an 'active plume' (the organized particle motion caused by buoyancy) around the plume centreline, and a 'passive plume' caused by turbulent dispersion of particles of the 'active plume'. The radius R of the 'active plume' is proportional to the mean plume height z over the source, $R = R_0 + \beta' z$, where β' (= 0.65) is the plume entrainment constant and R_0 is the initial plume radius in the bent-over stage. The mean plume height over the source is then given by

$$z = \left\{ \frac{3F_0}{\beta'^2 u(f^2 + N^2)} \cdot \left[1 - \left(\cos(Nt) + f/N \cdot \sin(Nt) \right) \cdot e^{-ft} \right] + \left(\frac{R_0}{\beta} \right)^3 \right\}^{1/3} - \frac{R_0}{\beta}, \tag{8}$$

where the contribution related to the initial momentum of the plume is neglected in comparison with the buoyancy. Here, F_0 is the initial values of the plume buoyancy, $N^2 = \beta g \, \partial\Theta / \partial x^3$ and f is a turbulence buffet frequency. The mean horizontal wind velocity u is written here small, because it is proposed to be constant within this approach.

This theory calculates the mean buoyant plume rise in dependence on the turbulence intensity and for arbitrary stratified flows. It provides the "two-thirds" power law,[9] since (8) becomes for a neutral stratification (N = 0), a laminar wind (f = 0) and a vanishing initial plume radius $R_0 = 0$,

$$z = \left\{ \frac{3F_0}{2\beta'^2 u} \right\}^{1/3} \cdot t^{2/3}. \tag{9}$$

The essential achievement of this theory is the explanation of the levelling off of the plume as result of extrainment, i.e. entrainment of plume material into the surrounding fluid

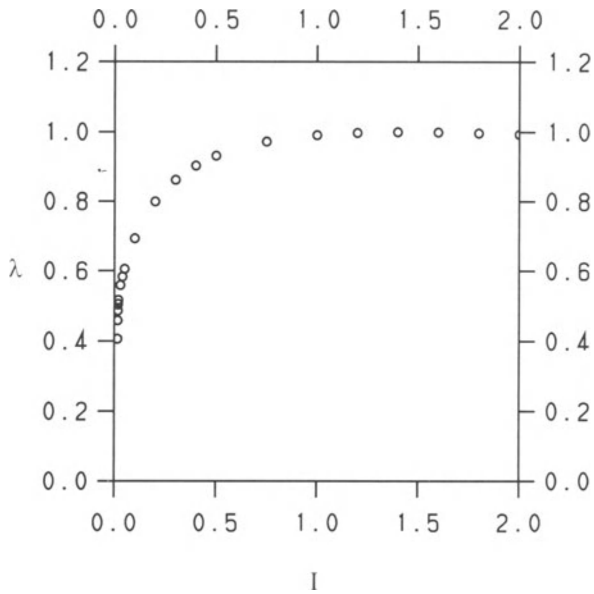

Figure 2. The factor λ in equation (13) in dependence on the normalized initial time scale.

due to the ambient turbulence. This is handled by the introduction of a turbulence buffet frequency f. This quantity is defined by $f = 2\ \beta'\ i_E\ u\ /\ \Lambda_E$, where i_E is the intensity of turbulence and Λ_E is the length scale of large-scale eddies. Neglecting again R_0, the final plume rise z is determined by (8) as

$$z = \left\{ \frac{3F_0}{\beta'^2\ u(f^2 + N^2)} \right\}^{1/3}. \tag{10}$$

Let us compare now the calculation of the plume behaviour in the Lagrangian and Eulerian approach in a non-turbulent and turbulent flow, i.e. with respect to the prediction of the "two-thirds" power law and the final plume rise, respectively. The Lagrangian approach provides the observed power law, if the relation (7) is satisfied. By adopting the definition of I it may be seen, that also the coefficients of the curves (6) and (9) are equal, when the initial particle time scale is given by

$$\tau_0 = \frac{1}{\sqrt{\beta'\beta^*}} \cdot \left(\frac{F_0}{B_0{}^3 u} \right)^{1/4}, \tag{11}$$

where

$$\frac{1}{\sqrt{\beta^*}} = \sqrt{\frac{2}{3}} \cdot \left(C_{\varepsilon 2} - 1\right) \cdot \left(1 + \frac{k_1(1 - k_3/k_1)}{2\left(C_{\varepsilon 2} - 1\right)}\right)^{3/4} \tag{12}$$

is introduced. With the above applied parameter values one finds $\beta^* = 0.65$ corresponding with the value for β applied by Netterville. For a turbulent flow, the final rise in the Lagrangian approach can be estimated numerically and is given by

$$\langle x_L{}^3 \rangle = 1.838 \cdot I \cdot \lambda(I) \cdot \frac{B_0}{\left(\partial U / \partial x^3\right)^2}, \tag{13}$$

where the curve $\lambda(I)$ is presented in Figure 8. By comparing this expression with (10) one finds, that both approaches provide the same final value, if

$$f = \beta^* \eta \cdot \frac{\partial U}{\partial x^3}, \qquad (14)$$

where $\eta = 0.7\, I^{1/2}\, \lambda(I)^{-3/2}$. Consequently, by the relations (11) and (14) constraints are given for the essential parameters which determine entrainment and extrainment in the Eulerian approach, β and f. B_0 is relatively easy to derive from measurements, but this is not the case for the initial particle time scale τ_0. This problem can be avoided by applying the relation (11) as definition for τ_0. On the other hand, the relation (14) combined with (11) for the calculation of I is very helpful, because f is explained in this way by measurable quantities. Instead, this quantity is given in the Eulerian approach by $f = 2\, \beta'\, i_E\, u / \Lambda_E$, where e.g. the intensity of turbulence i_E can hardly be derived from measurements. This requires then different ad hoc assumptions in order to estimate f which are debatable.[7] Moreover, the combination of (14) with (10) represent then a simple method to calculate the final plume rise, for which a lot of different empirical formulas exist.

CONCLUDING REMARKS

The Eulerian concept provides a phenomenological theory for the buoyant plume rise in accord with measured data. It can be applied under complex conditions, i.e. for shear flows with an arbitrary stratification.[1-3] Stochastic Lagrangian models provide firstly insight into the calculation of parameters applied in the Eulerian approach as shown with respect to the estimation of the turbulence buffet frequency f, and secondly, this approach solves simultaneously the dispersion problem, that means the plume width can be estimated in contrast to the Eulerian approach, where only the radius of the 'active plume' is obtained.

By the results presented here, the extrainment idea of Netterville for a turbulent flow is proved. This permits a simple calculation of the final plume rise by means of quantities, which can be measured directly. For non-turbulent flow, the similarity behaviour of the buoyant plume rise is explained by stochastic particle motion and a frequency change according to Kolmogorov's frequency equation.

Further applications of this Lagrangian approach to plume rise simulation are in preparation. These investigations are aimed at the demonstration of the advantages for the calculation of reactive plumes by the approach to describe the turbulent mixing.

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