ON THE INFLUENCE OF NON-GAUSSIANITY ON TURBULENT TRANSPORT

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Abstract. Non-Gaussianity effects, first of all the influence of the third and fourth moments of the velocity probability density function, have to be assessed for higher-order closure models of turbulence and Lagrangian modelling of turbulent dispersion in complex flows. Whereas the role and the effects of the third moments are relatively well understood as essential for the explanation of specific observed features of the fully developed convective boundary layer, there are indications that the fourth moments may also be important, but little is known about these moments. Therefore, the effects of non-Gaussianity are considered for the turbulent motion of particles in non-neutral flows without fully developed convection, where the influence of the fourth moments may be expected to be particularly essential. The transport properties of these flows can be characterized by a diffusion coefficient which reflects these effects. It is shown, for different vertical velocity distributions, that the intensity of turbulent transport may be enhanced remarkably by non-Gaussianity. The diffusion coefficient is given as a modification of the Gaussian diffusivity, and this modifying factor is found to be determined to a very good approximation by the normalized fourth moment of the vertical velocity distribution function. This provides better insight into the effect of fourth moments and explains the varying importance of third and fourth moments in different flows.

1. Introduction

A Gaussian shape is often assumed for the distribution of turbulent velocity fluctuations. This probability density function (pdf) is applied as a satisfactory approximation in the theory of homogeneous turbulence (Batchelor, 1953). In the atmosphere it is found, for instance, in the surface layer under neutral stratification (Du *et al.*, 1994a). Deviations of the pdf from the Gaussian shape occur, for example, when the spatial transport of turbulent kinetic energy (TKE) contributes to the structure of turbulence. This can be seen from the budget equation of TKE (Stull, 1988), where the spatial gradients of third moments of the velocity pdf determine the turbulent transport of TKE. These transports are especially important for the structure of the convective boundary layer. The non-Gaussianity of the velocity pdf under these conditions is demonstrated in convection tank experiments (Deardorff and Willis, 1985), by field experiments (Taconet and Weil, 1982; Caughey *et al.*, 1983) and by large-eddy simulations (Moeng and Wyngaard, 1989; Schmidt and Schumann, 1989).

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An assessment of such non-Gaussianity effects is important for turbulence modelling. The Eulerian approach leads to the well-known hierarchy of coupled transport equations for the moments of the velocity pdf. This leads to the need to derive closed equations for the moments of lower order, where often second-order closures are used providing closed variance equations (Mellor and Yamada, 1982). As stated by Stull (1988) it is generally assumed that equations for lower-order variables become more accurate as the closure assumptions are pushed to higher orders. There are different attempts to derive parametrizations of fourth-order terms, but very little is known about these moments and there is little guidance for suggesting good parametrizations. Usually, they are supposed to be quasi-Gaussian and written as functions of the second moments (references can be found in Stull, 1988). The Lagrangian modelling of the motion of fluid particles (Monin and Yaglom, 1971, 1975) represents an alternative approach to the description of turbulence and shows very attractive conceptual features (Pope, 1994a). This approach has been successfully applied to study the realizability of solutions of second-order models (Durbin and Speziale, 1994; Pope, 1994b) and to calculate turbulent dispersion of passive tracers in complex flows (van Dop *et al.*, 1985; Thomson, 1987; Sawford, 1993). But these particle models have to be chosen with a dependence on the shape of the Eulerian velocity pdf to ensure consistency between the Eulerian and the Lagrangian view (Thomson, 1987). For real flows only partial information is available for the estimation of this function in terms of the moments of lower order. For given third and fourth moments Du *et al.* (1994a,b) showed, for instance, how a maximum missing information pdf can be constructed. It is interesting to note that this approach requires given third and fourth moments (the number of considered moments has to be even). This leads to the problem of finding appropriate values for these moments.

Consequently, the role and the effect of the third as well as the fourth moments of the velocity pdf have to be investigated. This is relatively well understood with respect to the third moments. These terms explain the turbulent transport of TKE in the updrafts and downdrafts in the fully convective boundary layer (Hunt *et al.*, 1988). The effect of these terms is essential (Baerentsen and Berkowicz, 1984; de Baas and Troen, 1989; Luhar and Britter, 1989) to explaining the features for the mean particle height and vertical particle spread which are found in the water tank experiments of Willis and Deardorff (1976, 1978, 1981). In contrast, there is not a comparable understanding of the fourth moments, which may in particular be expected to characterize the turbulence structure in non-neutral, non-fully convective flows.

To gain increased insight into the significance of these moments, the effect of non-Gaussian distributed velocities on the motion of fluid particles is considered here for flows which show no strongly developed convective structures and may be non-neutral. It is convenient to investigate this by considering changes of the diffusion coefficient (K) caused by this non-Gaussianity. For approximately homogeneous and stationary turbulence K reflects the time integral of the velocity correlation and the time behaviour of the mean squared particle distance is governed by this coefficient for large times (Seinfeld, 1986). In the Eulerian view the diffusion coefficient, together with the mean wind, determines the change of concentration of a passive tracer. The calculation of the diffusion coefficient from stochastic fluid particle motion theory is considered in the next section. Adopting the Kolmogorov (1942) approximation for the dissipation, the symmetric component of the diffusion coefficient is then obtained from a dependence on the velocity pdf and on the mean dissipation rate of TKE. By considering different models of vertical velocity pdf's the modifications of the diffusion coefficient are investigated in Sections 3 and 4. The conclusions concerning the role and effect of third and fourth moments are summarized in the last section.

2. The Diffusion Coefficient

An equation of turbulent diffusion can be obtained if the hydrodynamic budget equation for the concentration of a passive tracer is closed by a mixing-length model (Seinfeld, 1986). The range of applicability of this equation can be made plausible in this way, but there is little guidance to obtain assertions on the diffusion coefficient K under inhomogeneous conditions. A derivation of this equation and a simultaneous calculation of K needs a theory for fluid particle motion which is the concern of the Lagrangian approach (Sawford, 1993; Pope, 1994a). In this framework a turbulent flow is considered as a continuum of fluid particles. Each particle has a constant mass, a position and velocity, and possibly properties such as a potential temperature or a chemical composition. In a high-Reynolds number turbulent flow this particle motion can be described by a Fokker–Planck equation for the position-velocity pdf from which, by elimination of the velocity, the diffusion equation can be derived as the Fokker–Planck equation for the position pdf. By this procedure Thomson (1987) derived the diffusion equation for the change in time t and space $\mathbf{x} = (x^1, x^2, x^3)$ of the ensemble averaged (symbol $\langle\cdots\rangle$) concentration $\langle c\rangle$ under a condition for the dissipation of TKE such that strongly developed convective conditions are excluded. This leads to the equation,

$$\frac{\partial\langle c\rangle}{\partial t} + \frac{\partial}{\partial x^i}[\bar{U}^i \cdot \langle c\rangle] = \frac{\partial}{\partial x^i}\left\{\langle\rho\rangle K^{ij}\frac{\partial}{\partial x^j}\left[\frac{\langle c\rangle}{\langle\rho\rangle}\right]\right\}. \tag{1}$$

Here, summation over repeated superscripts is assumed, $\bar{\mathbf{U}} = \langle\rho\rangle^{-1}\,\langle\rho\mathbf{U}_E\rangle$ is written for the density-weighted mean Eulerian (subscript E) velocity vector with components $\bar{U}^i$ ($i = 1, 2, 3$), $\langle\rho\rangle$ is the mean mass density of the flow and K is the diffusion coefficient matrix with elements K^{ij}. The mean concentration $\langle c\rangle$ is determined by the positions of M marked particles (δ is the delta function),

$$\langle c(\mathbf{x},t)\rangle = dm\sum_{n=1}^{M}\langle\delta(\mathbf{x} - \mathbf{x}_L^{(n)}(t))\rangle, \tag{2}$$

where dm is the constant mass of a particle and the position of the nth particle is denoted by the superscript (n) and the subscript L indicates a Lagrangian quantity. By the derivation of this equation from the stochastic theory of particle motion, in particular the diffusion coefficient K can be estimated in terms of particle properties. As shown below, the well-known Gaussian expression for the diffusion coefficient arises from the symmetric component $K_s = [K + K^T]/2$ of K (the superscript T denotes the transposed matrix). Because we are interested in deviations of the diffusion coefficient from the Gaussian expression due to non-Gaussianity of the velocity pdf, let us consider now the component K_s. This depends on the Eulerian pdf F for velocities $\mathbf{U}$ at given locations $\mathbf{x}$ and times t which is determined by

$$F(\mathbf{U}, \mathbf{x}, t) = dm \sum_{n=1}^{M_{\text{tot}}} \langle \delta[\mathbf{U} - \mathbf{U}_L^{(n)}(t)] \delta[\mathbf{x} - \mathbf{x}_L^{(n)}(t)] \rangle, \tag{3}$$

where the velocities $U_L^{(n)}(t)$ and positions $x_L^{(n)}(t)$ of all particles of the flow contribute to the sum (M_{tot} is the total number of particles in contrast to M). This function F is normalized to the mean mass density $\langle \rho \rangle$, i.e. it obeys

$$\int \mathrm{d}\mathbf{U} F(\mathbf{U}, \mathbf{x}, t) = \langle \rho(\mathbf{x}, t) \rangle. \tag{4}$$

Secondly, K_s depends on a quantity B which determines the intensity of the stochastic component of particle motion caused by the small-scale components of turbulence. Adopting the approximation of Kolmogorov (1942) for the dissipation of TKE, this quantity is independent of the velocity and given by (Thomson, 1987)

$$B^{ij} = (1/2) C_0 \langle \varepsilon \rangle \delta_{ij}, \tag{5}$$

where $\langle \varepsilon \rangle$ is the mean dissipation rate of TKE, δ_{ij} is the Kronecker delta and C_0 is an unknown parameter, for which a wide range of estimates is to be found in the literature. A good agreement between model predictions with laboratory measurements and observations in the atmospheric boundary layer was found for values $2.0 < C_0 < 3.5$ (Du *et al.*, 1995 and Heinz, 1997). From these two quantities, F and B, the symmetric component of the diffusion coefficent is given by

$$K_s^{ij} = \langle \rho \rangle^{-1} \langle \rho (\tilde{V} B^{-1} \tilde{V}^T)^{ij} \rangle = \langle \rho \rangle^{-1} \int \mathrm{d}\mathbf{U} F (\tilde{V} B^{-1} \tilde{V}^T)^{ij}, \tag{6}$$

where the quantity $\tilde{V}^{ij}$ satisfies the equation,

$$-\frac{\partial}{\partial U^k} [F \tilde{V}^{ik}] = (U^i - \bar{U}^i) F, \tag{7}$$

and the condition $\int \mathrm{d}\mathbf{S}^i F\tilde{V}^{ij} = 0$, where $\mathrm{d}\mathbf{S}$ is an element of the surface as $|\mathbf{U}| \to \infty$. This quantity $\tilde{V}^{ij}$ is completely determined by the distribution of velocity fluctuations and its mean value is given by

$$\langle \tilde{V}^{ij} \rangle = \langle \rho \rangle^{-1} \int \mathrm{d}\mathbf{U} F \tilde{V}^{ij} = V^{ij}, \tag{8}$$

where V^{ij} is an element of the matrix V of Eulerian second-order moments with

$$V^{ij} = \langle \rho \rangle^{-1} \langle \rho (U_E^i - \bar{U}^i)(U_E^j - \bar{U}^j) \rangle. \tag{9}$$

Let us now consider non-Gaussianity of only the vertical velocity pdf to simplify matters. For this we set $F = [F^G g(U^3)]/g_G(U^3)$, where F^G is a locally Gaussian pdf with (det V being the determinant of the second-moment matrix V)

$$F^G = \frac{\langle \rho \rangle}{(2\pi)^{3/2} (\det V)^{1/2}} \exp\left\{ -\frac{1}{2} \{U^i - \bar{U}^i\} (V^{-1})^{ij} \{U^j - \bar{U}^j\} \right\}, \tag{10}$$

g_G is the pdf of Gaussian-distributed vertical velocities which is obtained from F^G by integration over the horizontal velocity components,

$$g_G = \langle \rho \rangle^{-1} \int \mathrm{d}U^1 \mathrm{d}U^2 F^G = \frac{1}{(2\pi V^{33})^{1/2}} \exp\left\{ -\frac{1}{2} (V^{33})^{-1} (U^3 - \bar{U}^3)^2 \right\}, \tag{11}$$

and $g(U^3)$ is an unknown vertical velocity pdf. Deviations of $\tilde{V}$ from its mean value V appear then only in the element $\tilde{V}^{33}$, since $\tilde{V}^{ij} = V^{ij}$ for all $(i, j) \neq (3, 3)$. $\tilde{V}^{33}$ is obtained from (7) by

$$\tilde{V}^{33}(U^3) = -\frac{1}{g} \int_{-\infty}^{U^3} \mathrm{d}\hat{U}^3 (\hat{U}^3 - \bar{U}^3) g(\hat{U}^3). \tag{12}$$

Adopting (5) we then find the symmetric component K_s to be determined by the simple expression

$$K_s = B^{-1} V^2 \Gamma, \tag{13}$$

where Γ is a 3-dimensional matrix that modifies the Gaussian diffusion coefficient $B^{-1}V^2$ which follows for $g = g_G$. The elements of this matrix are given by

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \langle (\tilde{V}^{33})^2 \rangle / (V^{33})^2 \end{pmatrix}. \tag{14}$$

Consequently we see from (13) and (14) that a non-Gaussianity of the vertical velocity fluctuations is reflected in deviations Γ in the symmetric component K_s

of the diffusion coefficient from the Gaussian expression $B^{-1}V^2$. By this non-Gaussianity in the velocity pdf flow structures may be indicated (see below), whereby strong convective flows for which the diffusion limit breaks down (Garratt, 1992) are excluded. We note that these non-Gaussianity influences are reflected by the scalar Γ^{33}. This quantity Γ^{33} is investigated in the next two sections for different velocity pdf's.

3. Bimodal Non-Gaussianity

Investigation of the effect of the vertical velocity pdf g on Γ^{33} requires the construction of g in dependence on a few parameters. One way to do this uses the approach of Du *et al.* (1994a,b), where a maximum missing information pdf can be constructed based on an even number of given moments of this pdf. Assuming the knowledge of N moments, this pdf reads

$$g = \exp\left\{ - \sum_{k=0}^{N} \lambda_k (U^3)^k \right\},$$

where the λ_k can be calculated numerically from the given N moments. If moments up to fourth-order are taken into account in this maximum missing information pdf there is the problem to consider changes of the third and fourth moments which produce structures of practical relevance in the vertical velocity pdf. On the other hand it can be expected that the influence of non-Gaussianity on Γ^{33} emerges most clearly if the consequences of changing structures appearing in g are investigated. Therefore the vertical motion is considered at first as a superposition of two different processes just like updrafts and downdrafts in convective turbulence. For this, the vertical velocity pdf is taken as the sum of two Gaussian pdf's as assumed by Baerentsen and Berkovics (1984). Such a pdf has proved to reflect well the measured properties of convective flows (Luhar and Britter, 1989) and it has often been applied for the calculation of particle dispersion under these conditions (e.g. de Baas and Troen, 1989; Weil, 1990; Hurley and Physick, 1993). The vertical velocity pdf g multiplied by $(V^{33})^{1/2}$ is then given by

$$\begin{aligned}(V^{33})^{1/2} g = {} & \frac{a_-}{(2\pi)^{1/2}\sigma_-} \exp\left\{ -\frac{(w + w_-)^2}{2\sigma_-^2} \right\} \\ & + \frac{a_+}{(2\pi)^{1/2}\sigma_+} \exp\left\{ -\frac{(w - w_+)^2}{2\sigma_+^2} \right\}, \end{aligned} \tag{15}$$

where w denotes here and furthermore the vertical velocity U^3 normalized to $(V^{33})^{1/2}$, i.e. $w = U^3/(V^{33})^{1/2}$. Three conditions are given for the 6 parameters to be determined by the normalization, (i) $\int \mathrm{d}w[(V^{33})^{1/2} g] = 1$; (ii) the assumption of a vanishing mean vertical velocity, i.e. $\int dww[(V^{33})^{1/2}g] = 0$; and (iii) the consistency condition for the variances $\int dww^2[(V^{33})^{1/2}g] = 1$. To simplify matters it is supposed that the mean velocity w_+ of updrafts is converted into the turbulent energy represented by σ_+^2, that means $w_+ = \sigma_+$, and, corresponding for the sinking air, $w_- = \sigma_-$ (Baerentsen and Berkovics, 1984). This assumption was proved to be satisfactory by Luhar and Britter (1989). The ratio of the absolute values of the mean velocities in both modes will be defined now by $\gamma = w_+/w_-$. For convective turbulence this ratio is related to the fractional areas of rising and sinking motions. If A_u is the proportion of the area of updrafts and $1 - A_u$ is the proportion of the area of downdrafts, we find $\gamma = (1 - A_u)/A_u$. By Hunt *et al.* (1988) this ratio was found to be $\gamma = 1.5$ and the data of Lenschow and Stephens (1980) lead to $\gamma = 3.0$ as derived by Luhar and Britter (1989). Findings of A_u (Randall *et al.*, 1992) show that γ can be expected in a maximum range of $0.4 < \gamma < 5.3$. If γ is given, then all parameters are determined by it, where $a_+ = (1+\gamma)^{-1}$, $a_- = \gamma(1+\gamma)^{-1}$, $\sigma_+^2 = \gamma/2$, $\sigma_-^2 = 1/(2\gamma)$, and the skewness is determined by $s^3 = \langle w^3 \rangle = (\gamma - 1) \cdot (2/\gamma)^{1/2}$ and the kurtosis by $\mathrm{Ku} = \langle w^4 \rangle = (5/2)[1 + \gamma^{-1}(\gamma - 1)^2]$. The model (15) of vertical velocity fluctuations will be considered now for varying values of γ. A closed parameter set for this model was derived in the applications mentioned above using the relations of γ with the skewness (Baerentsen and Berkovics, 1984) and kurtosis (Du *et al.*, 1994b) and adopting height profiles for s^3 or the value Ku = 3, respectively. The introduction of γ has the advantage that the effect of varying third and fourth moments on Γ^{33} can be studied such that the considered basic structure of the model is maintained. In Figure 1 different vertical distribution functions and in Figure 2 the skewness and the kurtosis are shown as functions of γ. The calculated Γ^{33} is depicted in Figure 3, where for comparison the γ-dependence of Ku/2.5 is also shown. This normalization factor 2.5 represents the minimum of Ku(γ) as can be seen from Figure 2. This shows that a good approximation (the deviation is lower than 10%) to Γ^{33} is given by

$$\Gamma^{33} = \frac{\mathrm{Ku}}{\mathrm{Ku}(\gamma = 1)}, \tag{16}$$

where Ku(γ = 1) = 2.5 in this model. Let us consider a similar second model, where the two modes are themselves non-Gaussian. Here the vertical velocity pdf is assumed to be

$$\begin{aligned}(V^{33})^{1/2}g &= \frac{c_1 \exp(-a_1 w)}{a_1^{-1} - (a_1 + b_1)^{-1}}[1 - \exp(-b_1 w)] \\ &+ \frac{c_3}{(2\pi\sigma^2)^{1/2}} \exp(-w^2/(2\sigma^2))\end{aligned} \tag{17a}$$

for $w > 0$; for $w < 0$ this function is given by

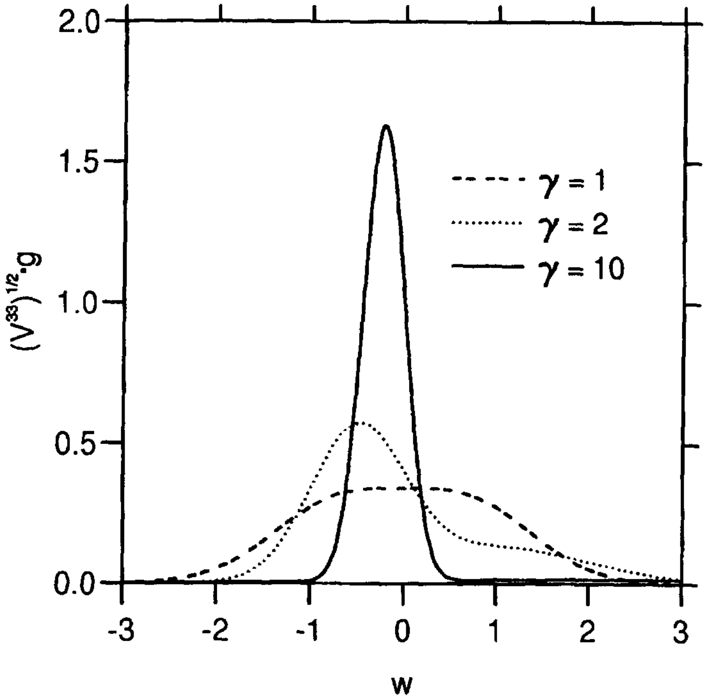

Figure 1. Three different distribution function densities of vertical velocity fluctuations in dependence on the ratio $\gamma = w_+/w_-$ of mean velocities w_+ and w_- of the positive and negative modes of the distribution function densities, respectively. The two modes are Gaussian.

$$(V^{33})^{1/2} g = \frac{c_2 \exp(a_2 w)}{a_2^{-1} - (a_2 + b_2)^{-1}}[1 - \exp(b_2 w)] + \frac{c_3}{(2\pi\sigma^2)^{1/2}} \exp(-w^2/(2\sigma^2)). \quad (17b)$$

For the calculation of the 8 parameters to be determined, the normalization condition and the consistency conditions have to be fulfilled in order to produce a first moment equal to zero and a second moment equal to V^{33}. Moreover, it is assumed to fit the contribution of the Gaussian function at $w = 0$, that the function value at $w = 0$ represents the mean value of the two mode peaks at positive and negative velocities and that the dispersion σ of this contribution is $(w_m^+ + w_m^-)/10$, where w_m^+ and w_m^- are the absolute values of the velocities, where the two modes have a maximum. The construction of this Gaussian contribution is just a procedure to ensure a smooth transition between the two non-Gaussian modes. Then 3 free parameters remain open. To close the parameter set, the ratios $\gamma_1 = b_1/a_1$, $\gamma_2 = b_2/a_2$ and $\gamma = (a_2/a_1)(1 + (1 + \gamma_1)^{-1})/[1 + (1 + \gamma_2)^{-1}]$ are introduced, where in particular γ gives the ratio of the mean velocities in the two modes. Consequent-

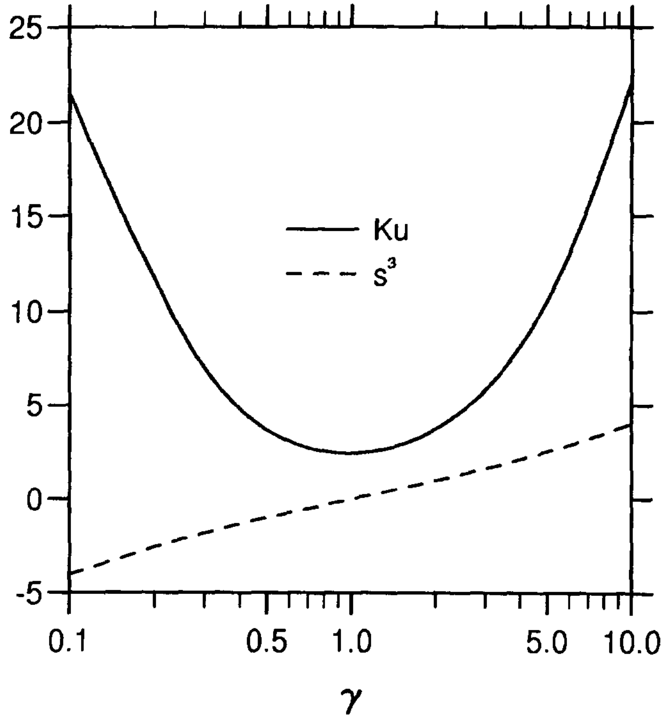

Figure 2. The skewness $s^3 = \langle w^3 \rangle$ and the kurtosis $\mathrm{Ku} = \langle w^4 \rangle$ of the vertical velocity pdf as functions of the mode ratio γ.

ly, all parameters and the skewness s^3 and the kurtosis Ku can be determined as functions of γ_1, γ_2 and γ as given in the appendix. The γ-dependence of the vertical velocity pdf as well as skewness and kurtosis are shown in Figures 4 and 5. The distribution functions and kurtosis and skewness show features which are similar to those obtained in the first model, but the maximum value of the pdf g for $\gamma = 1$ is considerably higher. In Figure 6 the γ-dependence of Γ^{33} is again compared with that of the normalized fourth moment, where the normalization Ku = 4 represents the minimum of $\mathrm{Ku}(\gamma, \gamma_1 = 1, \gamma_2 = 1)$ according to Figure 5. To investigate the influence of γ_1 and γ_2, in Figure 7 the latter one only is fixed ($\gamma_2 = 1$). This figure shows that the influence of γ_1 is very weak and the same can be found for that of γ_2. Consequently, it again follows to a good approximation (with a deviation lower than 10%) that $\Gamma^{33} = \mathrm{Ku}/\mathrm{Ku}_{\mathrm{min}}$.

These results illustrate that the fourth moments characterize the enhancement of the diffusivity caused by structures in the vertical velocity pdf. This effect can be considered on the level of the diffusion equation by a modification of the Gaussian diffusivity, but it can be taken into account, too, on the level of the more general fluid particle motion theory from which the diffusion equation, and with it the diffusion coefficient, are obtained. This can be deduced by a modification

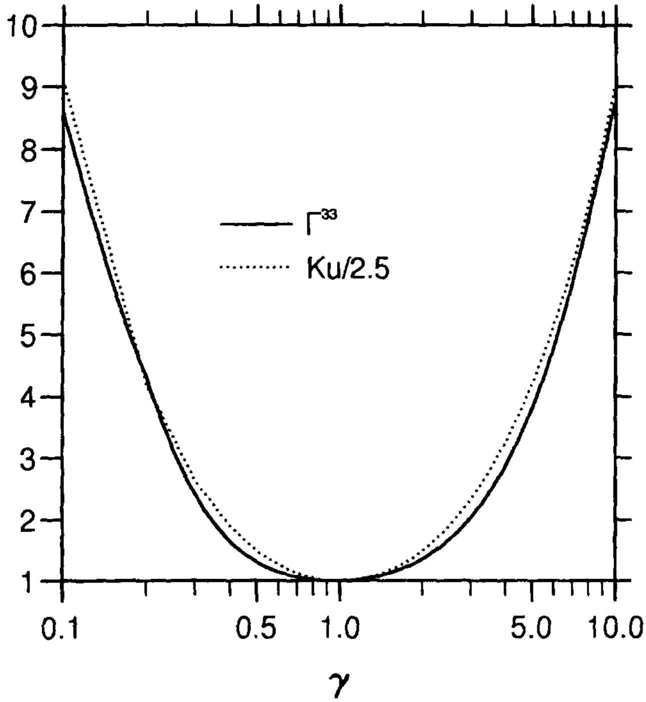

Figure 3. The dependence of the modification Γ^{33} of the Gaussian diffusion coefficient and of the normalized fourth moment Ku/2.5 on the mode ratio γ. The vertical velocity fluctuations are distributed according to Figure 1. The normalization factor 2.5 is the minimum of the fourth moment Ku as depicted in Figure 2.

of the turbulent time scale τ appearing e.g. in linear equations of motion (Heinz, 1997). For this, the symmetric component of the diffusion coefficient is written by (13) as $K_s^{33} = 4q^2\tau\Gamma^{33}/(9C_0)$, where (5) and $\langle\epsilon\rangle = q^2/(2\tau)$ are adopted, $q^2 = V^{kk}$ is twice the TKE, and for simplicity isotropic turbulence is assumed. For Gaussian turbulence this leads to $K_s^{33} = 4q^2\tau/(9C_0)$, and means that the influence of non-Gaussianity can be incorporated if τ is replaced in this expression by $\tau_{mod} = \tau\mathrm{Ku}/\mathrm{Ku}_{min}$. This leads then to an increase in the mean squared particle distance with time since $d\langle(x_L^3)^2\rangle/dt \sim \tau$ (Seinfeld, 1986).

4. Asymmetric Non-Gaussianity

The models considered above show that the non-Gaussian modification of the diffusivity can be explained over a wide range of γ by the behaviour of the fourth moments. To assess better the range of validity of this finding let us seek a model where the influence of the third moments is considerably larger than in the models

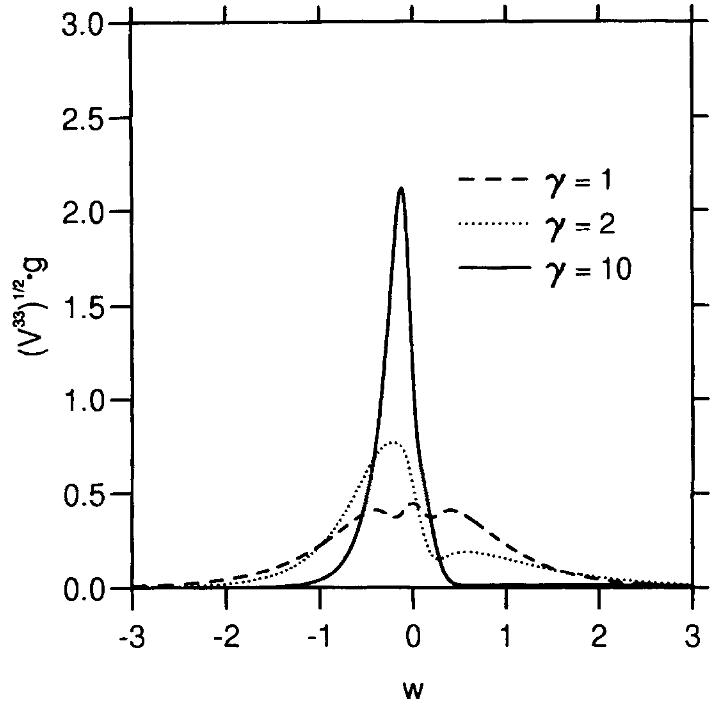

Figure 4. Three different distribution function densities of vertical velocity fluctuations in the model with two non-Gaussian modes, where γ is the ratio of mean velocities in the two modes. $\gamma_1 = \gamma_2 = 1$.

used above. For this we consider the general representation of g as a functional in its cumulants K_n (Risken, 1984),

$$(V^{33})^{1/2}g = \frac{1}{2\pi}\int \mathrm{d}u \exp(-iuw) \exp\left(\sum_{n=1}^{\infty} \frac{(iu)^n}{n!} K_n\right), \tag{18}$$

where w is normalized to $K_2^{1/2} = (V^{33})^{1/2}$ and the K_n (normalized to the corresponding powers of $K_2^{1/2}$) can be expressed by moments up to order n. By (12), $\tilde{V}^{33}$ is determined by

$$\frac{\tilde{V}^{33}}{V^{33}} = -\frac{1}{(V^{33})^{1/2}g}\int_{-\infty}^{w} \mathrm{d}\hat{w}(\hat{w} - K_1)(V^{33})^{1/2}g, \tag{19}$$

or applying (18) and integrating over $\hat{w}$, by

$$\frac{\tilde{V}^{33}}{V^{33}} = \frac{1}{(V^{33})^{1/2}g}\int \mathrm{d}u \frac{iu(w - K_1) + 1}{(iu)^2} \times \exp\left(-iu(w - K_1) + \frac{(iu)^2}{2}\right) \exp\left(\sum_{n=3}^{\infty} \frac{(iu)^n}{n!} K_n\right). \tag{20}$$

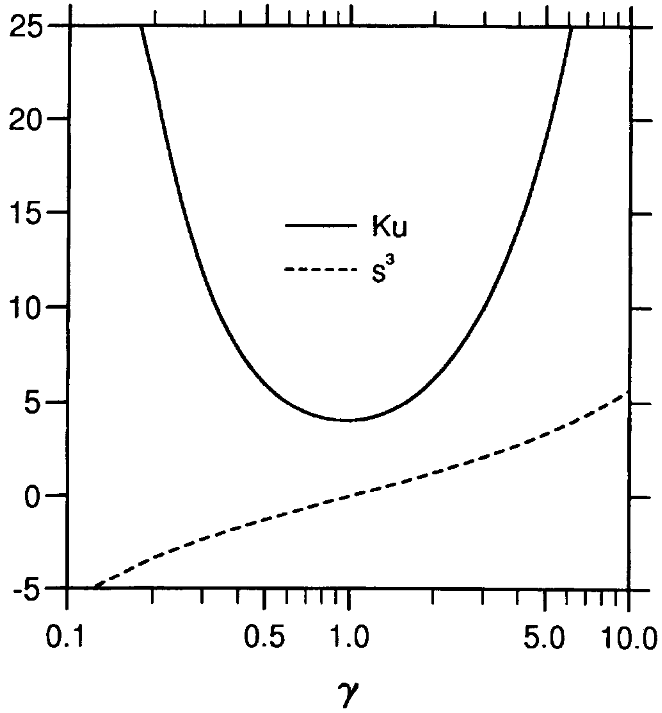

Figure 5. The skewness $s^3 = \langle w^3 \rangle$ and the kurtosis Ku = $\langle w^4 \rangle$ as in Figure 2 for the model with two non-Gaussian modes of vertical velocities as shown in Figure 4. These moments depend on the parameters γ_1 and γ_2 (appendix) which are set to unity.

This expression can be transformed by partial integration, leading to

$$\begin{aligned}\frac{\tilde{V}^{33}}{V^{33}} &= 1 - (w - K_1)\frac{g^{(1)}}{g} - (w - K_1)^2 \\ &+ \sum_{n-2}^{\infty} \frac{K_{n+1}}{n!}\left\{(-1)^n \frac{g^{(n)}}{g}(w - K_1) + (-1)^{n-1}\frac{g^{(n-1)}}{g}\right\}, \end{aligned} \qquad (21)$$

where $g^{(n)}$ denotes the nth derivative of g, i.e. $g^{(n)} = \partial^n g/\partial w^n$. To proceed further this expression has to be simplified. If the cumulants are assumed to vanish at some order ≥ 3, then oscillations of the pdf $(V^{33})^{1/2} g$ appear, where this function has positive as well as negative values (Risken, 1984). To ensure that g remains positive definite we take only the cumulants up to second-order into account. Then, it follows for Γ^{33} using (14)

$$\Gamma^{33} = \langle (w - K_1)^4 \rangle + \left\langle (w - K_1)^2 \left(\frac{g^{(1)}}{g}\right)^2 \right\rangle - 5. \qquad (22)$$

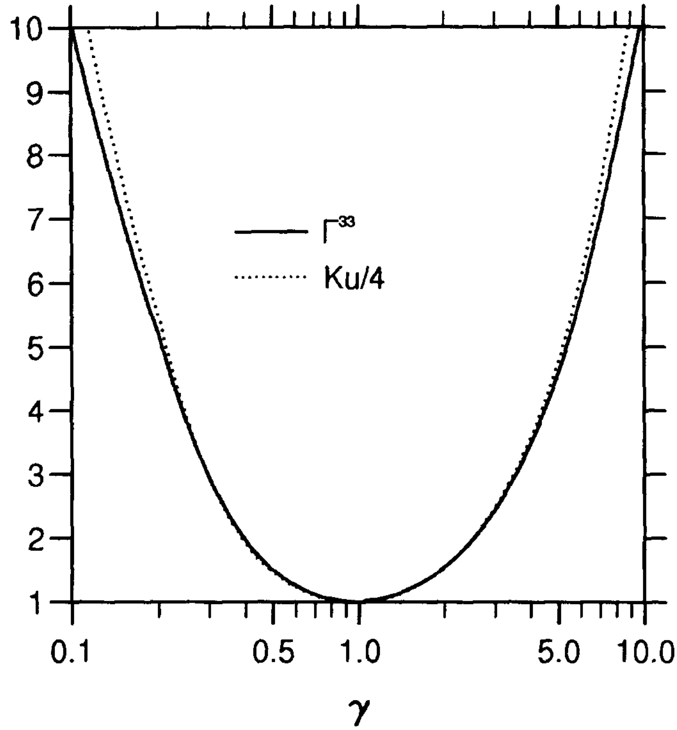

Figure 6. The dependence of the modification Γ^{33} of the Gaussian diffusion coefficient and of the normalized fourth moment Ku/4.0 on the mode ratio γ for the model with two non-Gaussian modes of the vertical velocity distribution (Figure 4). The normalization factor 4.0 is the minimum of the fourth moment Ku as depicted in Figure 5. $\gamma_1 = \gamma_2 = 1.0$.

This relation shows clearly the influence of the fourth moments on Γ^{33}. For approximately Gaussian turbulence with $g^{(1)}/g \approx -(w - K_1)$ we obtain

$$\Gamma^{33} \approx 2\langle(w - K_1)^4\rangle - 5, \tag{23}$$

which corresponds with the above derived $\Gamma^{33} = \mathrm{Ku}/\mathrm{Ku}_{\min}$. If $g^{(1)}/g$ is exactly Gaussian this leads to $\Gamma^{33} = 1$, since $\langle(w - K_1)^4\rangle = 3$. From (22) it can be seen that $g^{(1)}/g$ has to be at least a linear function in $(w - K_1)$ to ensure that third moments influence the modification of the diffusivity, that means $g^{(1)}/g \sim -(w - K_1 - s)$, where s is an unknown parameter. With the same arguments as in the section before, it is proposed now that $g^{(1)}/g$ is a superposition of linear functions in $-(w - K_1 - s)$ which can be seen as maximum missing information functions of second order ($N = 2$). If instead $N = 4$ is considered there is the problem to find values for the fourth moments Ku which harmonize with the third moments. Figures 2 and 5 show for instance that the fourth moments change, if the third moments vary. Accordingly, seeking $(V^{33})^{1/2}g$ as a polynomial in $v = (w - s)$

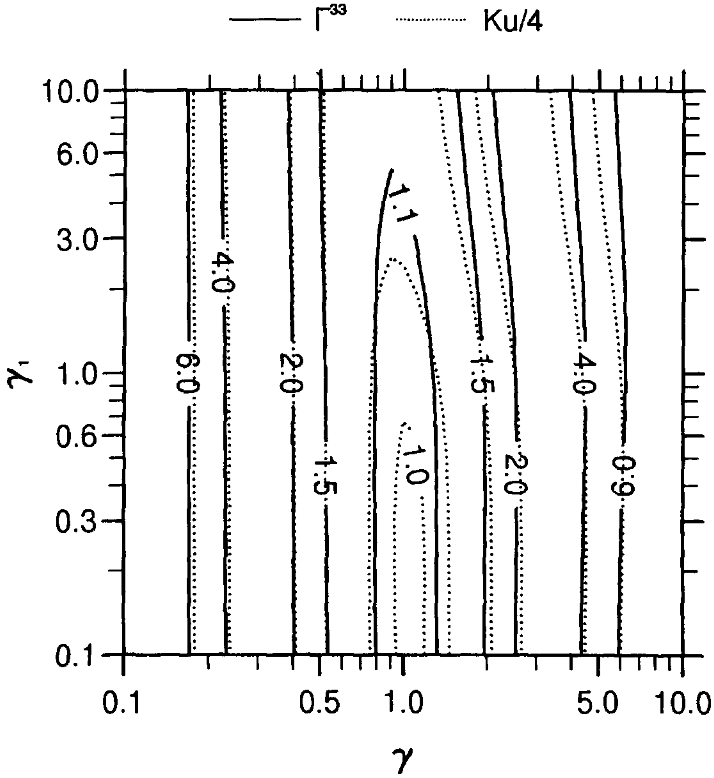

Figure 7. The modifying factor Γ^{33} and the normalized fourth moment Ku/4.0 in dependence on γ and γ_1. The parameter γ_2 has only an insignificant influence (as γ_1) and is set to unity.

multiplied by the assumed mode $\exp(-v^2/2)$, where the mean vertical velocity K_1 is proposed as zero as above, we deduce the model

$$(V^{33})^{1/2} g = \frac{1}{\sqrt{2\pi}} \left\{ \frac{s^2}{2} v^2 - sv + 1 - \frac{s^2}{2} \right\} \exp\left(-\frac{v^2}{2} \right), \tag{24}$$

which fulfils the normalization condition and guarantees a mean value of the vertical velocity equal to zero and a second moment equal to V^{33}. The same model was proposed by Thomson (1987) for the characterization of convective turbulence. It is very convenient for the investigation of the influence of the skewness, since the latter is related very simply to the parameter s by $s^3 = \langle w^3 \rangle$, and for the fourth moment it follows that $\langle w^4 \rangle = 3(1 + s^4)$. The shape of the pdf g is depicted for different values of s in Figure 8 (it has a much lower peak than the pdfs considered in the models above). The s-dependence of the skewness and the kurtosis is shown in Figure 9. Here we can notice in particular that the fourth moments are very near to 3 over a wide range of s. For Γ^{33} an analytical expression can be deduced which reads

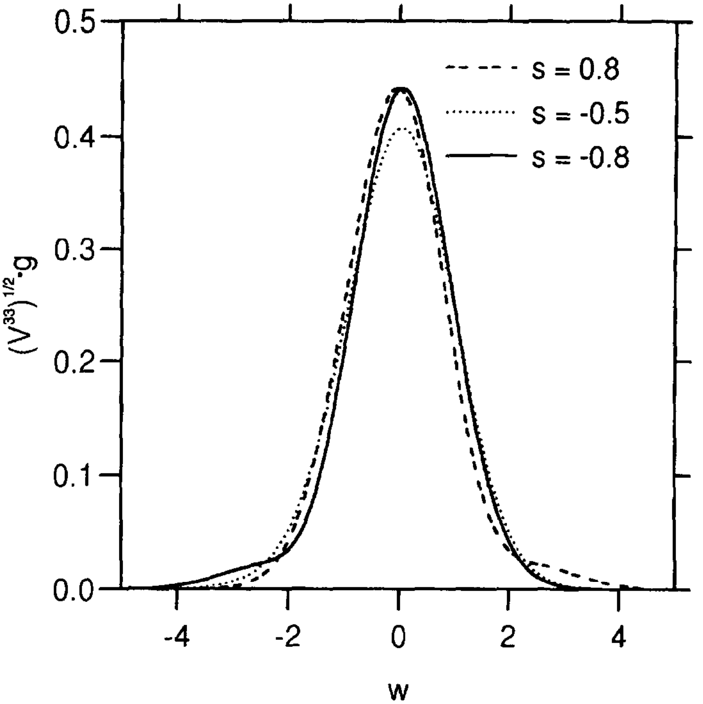

Figure 8. Three different distribution function densities of vertical velocity fluctuations in dependence on the skewness parameter s with $s^3 = \langle w^3 \rangle$ for the asymmetric non-Gaussian model.

$$\Gamma^{33} = 1 + \frac{s^4}{2}\left\{1 - 2x\sqrt{\pi}\,\mathrm{Im}w(x+iy) + \frac{x^2-y^2}{y}\sqrt{\pi}\,\mathrm{Re}w(x+iy)\right\}, \quad (25)$$

where w is a complex error function,

$$w(x) = \left\{1 + 2i\pi^{-1/2}\int_0^x dt\,\exp(t^2)\right\}\exp(-x^2). \quad (26)$$

Here, the abbreviations $x = 1/(2s^2)^{1/2}$ and $y = [(1/(2s^2)) - (1/2)]^{1/2}$ are used. The s-dependence of Γ^{33} is shown in Figure 10 compared with the s-dependence of the normalized fourth moment and $(1 + s^8)$ which fits the s-curve of Γ^{33} very well. The relative error $\Delta(s)$ of approximating $\Gamma^{33} \approx (1 + s^8)$ again by $\mathrm{Ku}/\mathrm{Ku}_{\mathrm{min}} = (1 + s^4)$ is $\Delta(s) = (1 + s^4)/(1 + s^8) - 1$. This error has a maximum of 0.207 at $|s| = 0.8$.

With these results obtained here and in the previous section it appears that the relation $\Gamma^{33} = \mathrm{Ku}/\mathrm{Ku}_{\mathrm{min}}$ is a fair approximation, but deviations up to 21% (a decrease of the diffusivity) can arise.

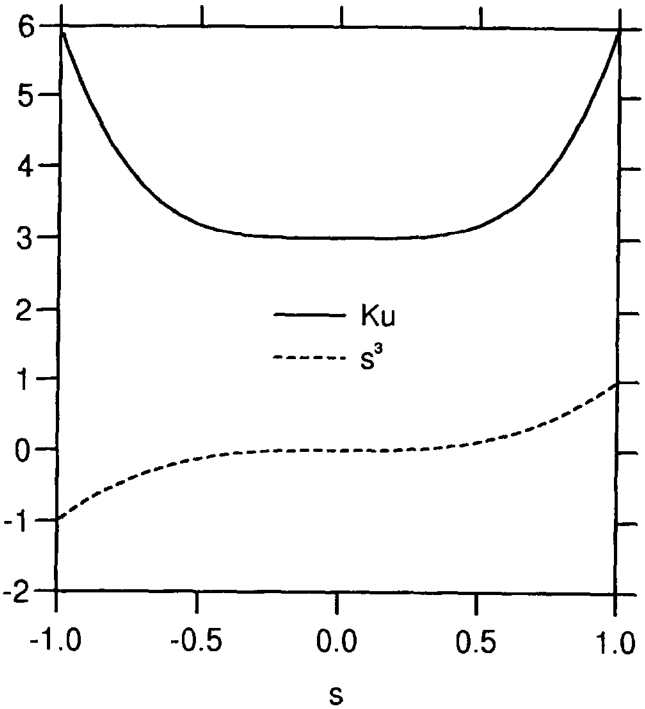

Figure 9. The skewness $s^3 = \langle w^3 \rangle$ and the kurtosis Ku = $\langle w^4 \rangle$ of the vertical velocity pdf given in Figure 8 in dependence on the skewness parameter s.

5. Concluding Remarks

These results show that the fourth moments of the velocity pdf characterize the enhancement of the Gaussian diffusivity caused by turbulence structures for the symmetric component of the diffusion coefficient. The expression $K_s = K_G \Gamma$ was found, where K_G is the diffusivity for Gaussian turbulence and Γ is a matrix arising from non-Gaussian distributed vertical velocity fluctuations. This quantity is equal to the unit matrix except for the element Γ^{33} which is determined to a good approximation by Γ^{33}=Ku/Ku$_{\text{min}}$, where Ku$_{\text{min}}$ is the minimum of the kurtosis Ku with respect to the parameters introduced in the different models to characterize the distribution function. The quantitative effect of this enhancement of the turbulent transport intensity can be remarkable. Within the first model e.g., a ratio of $\gamma = w_+/w_- = 3$ (which corresponds well with measurements as discussed in Section 3) between the mean-velocities in the two modes leads to a doubling of transport intensity. It is interesting to note that this influence of the fourth moments can be incorporated, too, at the level of the more general fluid particle motion theory as considered in Section 3. According to the findings obtained, the fourth moments are found to characterize the non-Gaussianity of velocity fluctuations essentially as long as the diffusivity concept works, that is as long as the TKE produced at some

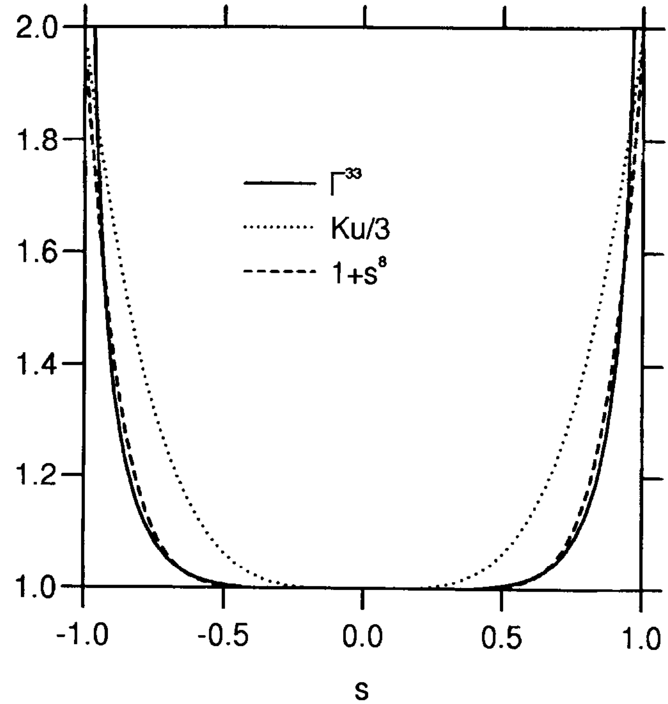

Figure 10. The dependence of the modification Γ^{33} of the Gaussian diffusion coefficient and of the normalized fourth moment Ku/3.0 on the skewness parameter s. The vertical velocity fluctuations are distributed according to Figure 8. The normalization factor 3.0 is the minimum of the fourth moment Ku in dependence on s as depicted in Figure 9. Additionally the curve $(1 + s^8)$ is shown which fits Γ^{33} very well.

location is removed mainly by dissipation and not by spatial transport. Deviations from the local TKE transfer can be taken into account by a non-Gaussianity of the velocity pdf (second section). As investigated in the fourth section, considering the maximum influence of the third moments on the modification of the Gaussian diffusivity, the enhancement of the transport intensity given by the fourth moments is diminished by the third moments but their influence is limited to a decrease of about 21%. As discussed in the introduction, the third moments emerge as important for the characterization of convective conditions, where the diffusivity concept breaks down.

These facts explain the role of the fourth moments for the description of the turbulence structure and support the approach of Du *et al.* (1994a, b) to the construction of the maximum missing information pdf which requires (for $N = 4$) knowledge of the third as well as of the fourth moments. However, there is the need to estimate convenient values of these quantities which is a difficult problem as can be seen from the data for Ku measured during unstable stratification and presented by Du *et al.* (1994b). These data show that Ku is found in a range $2.5 < \mathrm{Ku} <$

5 which corresponds well with the variations found here. For their simulations of particle dispersion a value Ku = 3 is derived from these measurements corresponding with Gaussian turbulence. This cannot be expected in general according to the result presented here. Instead it seems to be advantageous to seek an "expectation" value of Ku (the data of Du *et al.* support a value between 3.5 and 4.0) and to estimate a range of variations of this quantity. These estimations can be applied for instance to an improved prediction of the location of the maximum ground-level concentration of passive tracers which depends upon the value of Ku (Du *et al.*, 1994b).

On the Eulerian side, third-order closures require parametrizations for the fourth moments. For this, some guidance can be obtained from the first and third model used above (in the second model these functions in γ are more complicated). The fourth moments are related in the first model by $\langle\rho\rangle^{-1}\langle\rho(U_E^3 - \bar{U}^3)^4\rangle = (5/2)(V^{33})^2[1 + (1/2)\langle\rho\rangle^{-1}\langle\rho(U_E^3 - \bar{U}^3)^3\rangle^2/(V^{33})^3]$ with the second moment V^{33} given by (9) and the third moment, and in the third model this relation reads $\langle\rho\rangle^{-1}\langle\rho(U_E^3 - \bar{U}^3)^4\rangle = 3(V^{33})^2[1 + \langle\rho\rangle^{-1}\langle\rho(U_E^3 - \bar{U}^3)^3\rangle^{4/3}/(V^{33})^2]$. It is remarkable that no unknown parameters appear. A parametrization using the second moments only is given e.g. in the first model by $\langle\rho\rangle^{-1}\langle\rho(U_E^3 - \bar{U}^3)^4\rangle = (5/2)(V^{33})^2[1 + (\gamma - 1)^2/\gamma]$. The parameter γ can be expected to be in the maximum range $0.4 < \gamma < 5.3$ but probably it is found between 1.5 and 3.0 (Section 3). The assessment of the differences between these parametrizations and the choice of Ku within the maximum missing information pdf approach needs further investigation.

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Appendix

The parameters used in the bimodal Gaussian model with non-Gaussian modes (Section 3) can be calculated by means of the introduced ratios γ, γ_1 and γ_2. We first define,

$$A = \frac{1 + (1 + \gamma_1)^{-1}}{1 + (1 + \gamma_2)^{-1}}, \tag{A1}$$

$$B = \frac{1 - (1 + \gamma_1)^{-3}}{1 - (1 + \gamma_1)^{-1}}, \tag{A2}$$

$$C = \frac{1-(1+\gamma_2)^{-3}}{1-(1+\gamma_2)^{-1}}, \tag{A3}$$

$$D = 2\frac{B + A^2 C/\gamma}{1+\gamma}, \tag{A4}$$

$$E = \frac{(1+\gamma_1)^{-1/\gamma_1} + \gamma^2 A(1+\gamma_2)^{-1/\gamma_2}}{1+\gamma}, \tag{A5}$$

$$F = \frac{\ln(1+\gamma_1)}{\gamma_1} + \frac{A\ln(1+\gamma_2)}{\gamma\gamma_2}. \tag{A6}$$

Then $c_3 = (2\pi)^{1/2}EF/(20 + (2\pi)^{1/2}EF)$, $\sigma = (1 + 100D(1 - c_3)/F^2)^{-1/2}$, $a_1 = F/(10\sigma)$, $a_2 = \gamma a_1/A$, $b_1 = \gamma_1 a_1$, $b_2 = \gamma_2 a_2$, $c_1 = (1 - c_3)/(1 + \gamma)$ and $c_2 = \gamma(1 - c_3)/(1 + \gamma)$. The skewness and the kurtosis are given in terms of these parameters as

$$s^3 = 6\left[c_1\frac{a_1^{-4} - (a_1+b_1)^{-4}}{a_1^{-1} - (a_1+b_1)^{-1}} - c_2\frac{a_2^{-4} - (a_2+b_2)^{-4}}{a_2^{-1} - (a_2+b_2)^{-1}}\right], \tag{A7}$$

$$\mathrm{Ku} = 3c_3\sigma^4 + 24\left[c_1\frac{a_1^{-5} - (a_1+b_1)^{-5}}{a_1^{-1} - (a_1+b_1)^{-1}} + c_2\frac{a_2^{-5} - (a_2+b_2)^{-5}}{a_2^{-1} - (a_2+b_2)^{-1}}\right]. \tag{A8}$$

The γ dependence of these moments can be seen in Figure 5.

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