SUPPLEMENTARY MATERIAL TO:

On Mean Flow Universality of Turbulent Wall Flows. I. High Reynolds Number Flow Analysis

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The model development ensures that the mean velocity correctly agrees with the asymptotic velocities at the centerline/free-stream boundary. However, this does not necessarily imply that the velocity gradients disappear there, as required. With respect to S_1^+, Eq. (A.2) reveals that S_1^+ disappears for the TBL, but for channel and pipe flow there is at $y=1$ a very small but nonzero contribution

$$S_1^+(1) = 1 - \left[\frac{(Re_\tau/a)^{b/c}}{1+(Re_\tau/a)^{b/c}} \right]^c, \tag{S.1}$$

which becomes smaller with increasing Reynolds number. To ensure that S_1^+ is fully consistent with the requirement of zero gradients at the centerline, we introduce for channel and pipe flow a modified S_1^+ by

$$S_1^* = 1 - \left\{ (1-y) \left[\frac{(y^+/a)^{b/c}}{1+(y^+/a)^{b/c}} \right]^c + \frac{y}{1-S_1^+(1)} \left[\frac{(y^+/a)^{b/c}}{1+(y^+/a)^{b/c}} \right]^c \right\}. \tag{S.2}$$

This expression ensures a smooth transition of S_1^+ to its asymptotic value zero at $y=1$. By using Eq. (A.2), the latter equation can be written more conveniently as $S_1^* = S_1^+ + S_1^{CP}$, where

$$S_1^{CP} = -y S_1^+(1) \frac{1-S_1^+}{1-S_1^+(1)}. \tag{S.3}$$

The implied U_1^{CP} in the modified velocity $U_1^* = U_1^+ + U_1^{CP}$ follows from integrating Eq. (S.3) combined with the condition that the shear correction does not affect U_1^+ at $y=0$,

$$U_1^{CP} = \frac{S_1^+(1)}{1-S_1^+(1)} \, \frac{a^2}{2Re_\tau} \left[cB_G\left(c+\frac{2c}{b}, 1-\frac{2c}{b}\right) + G^{\frac{2c}{b}}(1-G)^{-\frac{2c}{b}} - G^{c+\frac{2c}{b}}(1-G)^{-\frac{2c}{b}} \right]$$
$$-\frac{y^{+2}}{2Re_\tau} \frac{S_1^+(1)}{1-S_1^+(1)}. \tag{S.4}$$

The maximum contribution due to U_1^{CP} can be evaluated in the following way. By using Eq. (A.2), we can prove that $1-S_1^+ \leq 1-S_1^+(1)$. Hence, $|S_1^{CP}| \leq y S_1^+(1)$. The integration of this expression then provides $|U_1^{CP}| \leq S_1^+(1) y^{+2}/(2Re_\tau)$. The maximum relative change of this contribution with respect to U_∞^+ is then given by $e = S_1^+(1) Re_\tau/(2U_\infty^+)$ at $y=1$. For the lowest Reynolds number considered here, $Re_\tau = 500$, this relative change is 0.27%, and this relative change rapidly becomes smaller with increasing Reynolds numbers. Such differences smaller than 0.27% do not produce any visible effects in the velocity figures shown here.

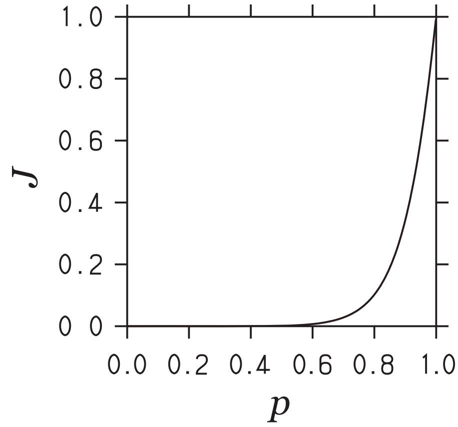

Figure S.1. $J(p)$ according to Eq. (S.9): the black line shows the numerical solution to Eq. (S.9). The pink line (which is hardly visible) shows the approximation Eq. (S.10).

With respect to S_2^+ and S_3^+, there is again no problem for the TBL where S_2^+ and S_3^+ balance each other asymptotically. For channel and pipe flow, S_2^+ and S_3^+ almost balance each other asymptotically, but there is a very small difference at $y = 1$ produced by small deviations of S_2^+ from $1/(\kappa y^+)$. To adjust this difference, we follow a similar approach as with respect to the adjustment of S_1^+. We have

$$S_2^+(1) = \frac{1}{\kappa Re_\tau} \frac{1 + h_3/[1 + Re_\tau/h_1]}{1 + y_\kappa/(Re_\tau H(y=1))}, \tag{S.5}$$

and we introduce a modification S_2^* of S_2^+ by

$$\kappa y^+ S_2^* = (1-y)\frac{1 + h_3/[1 + y^+/h_1]}{1 + y_\kappa/(y^+ H)} + \frac{y}{\kappa Re_\tau S_2^+(1)} \frac{1 + h_3/[1 + y^+/h_1]}{1 + y_\kappa/(y^+ H)}. \tag{S.6}$$

This correction ensures $\kappa y^+ S_2^+ = 1$ at $y = 1$, as needed for the balance of S_2^+ and S_3^+. A more convenient writing of the latter equation is given by $S_2^* = S_2^+ + S_2^{CP}$, where

$$S_2^{CP} = -y S_2^+ \left(1 - \left[\kappa Re_\tau S_2^+(1)\right]^{-1}\right). \tag{S.7}$$

This shear modification implies a corresponding outer boundary layer correction U_2^{CP} in $U_2^* = U_2^+ + U_2^{CP}$,

$$U_2^{CP} = -\left(1 - \left[\kappa Re_\tau S_2^+(1)\right]^{-1}\right) \frac{y}{\kappa} J(p). \tag{S.8}$$

This expression involves the abbreviation

$$J(p) = \frac{1-p}{p} \int_0^p \frac{s^{1+h_3}}{(1-s)^2} \frac{1 + h_3(1-s)}{s^{1+h_3} + (1-s) y_\kappa/h_1} ds. \tag{S.9}$$

This integral involves $p = y^+/h_1/[1 + y^+/h_1]$, which is bounded by zero and one, $0 \le p \le 1$. An exact analytical calculation of this integral is not possible (and it would be of limited usefulness because it would involve relatively complex functions like hypergeometric functions, as may be seen by solving the simplified integral over

$s^{1+h_3}/(1-s)^2$). A much more appropriate approach is to derive a highly accurate but simple analytical approximation. A very good approximation to $J(p)$ is given by $J(p) = p^{10}$. A better approximation is shown in Fig. S.1, which demonstrates the excellent performance of the approximation

$$J(p) = \alpha p^{1+h_3} \left[1 + (\alpha - 1)p + 1.236p(p-1)(p-0.55)\right]^{-1}, \tag{S.10}$$

where $\alpha = h_1(1+h_3)/[y_\kappa(2+h_3)]$. The term αp^{1+h_3} in front of the bracket arises from the initial scaling of $J(p)$. The bracket term describes an slightly modified linear function that varies between 1 (at $p = 0$) and α (at $p = 1$). The relative error of the largest deviation between $J(p)$ and the approximation Eq. (S.10) is below 0.4%: there is no visible difference in Fig. S.1. The relative impact of shear-induced corrections can be seen in the following way. From Eq. (A.12) we see that $yS_2^+ \leq S_2^+(1)$. This implies $|S_2^{CP}| \leq (1 - [\kappa Re_\tau S_2^+(1)]^{-1})S_2^+(1)$ according to Eq. (S.7). This expression implies by integration $|U_2^{CP}| \leq (1 - [\kappa Re_\tau S_2^+(1)]^{-1})S_2^+(1)y^+$. The maximum relative change of this contribution with respect to U_∞^+ is then $e = (1 - [\kappa Re_\tau S_2^+(1)]^{-1})S_2^+(1)Re_\tau/U_\infty^+$ at $y = 1$. For the lowest Re_τ considered here, $Re_\tau = 500$, this relative change is 0.22%, and this relative change rapidly becomes smaller with increasing Reynolds numbers. Such differences smaller than 0.22% do not produce any visible effects in the velocity figures shown here.