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The large eddy simulation capability of Reynolds-averaged Navier-Stokes equations: Analytical results

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ABSTRACT

A highly attractive idea to overcome shortcomings of both Reynolds-averaged Navier-Stokes (RANS) and large eddy simulation (LES) equations is the implementation of LES capability in RANS models. However, this approach faces questions regarding (i) the measurement of resolved motion and equivalence of various equations having the same resolution, (ii) the continuous variation of resolved and modeled motion under grid variations, and (iii) the explanation of how resolved motion and scaling variables in LES depend on the grid. Corresponding analytical results (addressing the resolution measurement, equivalence of equations, and resolution control) are reported, and grid effects (including the scaling of computational cost) are discussed.

The need for a significant cost reduction of large eddy simulation (LES), in particular for wall-bounded turbulent flows, initiated a relevant research topic that is under investigation over decades:[1–12] the hybridization of Reynolds-averaged Navier-Stokes (RANS) and LES. An ideal expectation for the performance of hybrid RANS-LES methods is their optimal performance on each affordable grid up to high Reynolds numbers: depending on the resolution supported by the grid, the amount of modeled motion smoothly increases if the grid becomes coarser. In this regard, a major issue of popular hybrid RANS-LES methods is the balance of modeled and resolved turbulent motions: modeled and resolved motions cannot be considered to be independent; the modeled motion has to respond to the amount of resolved motion to ensure an appropriate total motion (given by the sum of resolved and modeled motions). The latter issue is not just an academic problem. For example, it is known to be the reason for an incorrect reflection of wall physics (the logarithmic law of the wall cannot be obtained[13]).

From a more general viewpoint, there are several questions with respect to the mathematical foundations of hybrid RANS-LES and LES methods:

- Q1. Is there a mathematical explanation of (i) how the amount of resolved and modeled motion has to be measured and (ii) how various turbulence models can produce the same resolution?
- Q2. Is there mathematical proof that many RANS equations (i) can produce resolved motions so that (ii) modeled motions are continuously in balance with the actual resolved motion?
- Q3. Which ideas of how the grid (the filter width Δ) affects (i) the resolution and (ii) LES scale variables (the LES length scale L) are supported by mathematical-physical principles?

Regarding Q1, a very important and non-trivial question is how the resolution of LES can be evaluated,[14] and hybrid RANS-LES models need to get information about the actual resolution to enable an appropriate generation of modeled motion. A closely related question is how different turbulence models can be designed that have the same resolution. Regarding Q2, the view that RANS equations can produce resolved motions under certain conditions is supported by both applications and theoretical arguments.[15] By focusing on criteria for the equivalence of various hybrid methods, Friess *et al.*[10] have recently presented a technique to analyze relations between model coefficients of RANS equations and variables that reflect the amount of modeled motion [like the modeled-to-total turbulent kinetic energy ratio $k_+ = k/k_{tot}$, where $k_{tot} = k + k_{res}$ is the sum of modeled (k) and resolved (k_{res}) energies]. However, these relations were presented under rather restrictive conditions (see below), and they change depending on the assumptions made. So there is certainly the question of what proves, in general, the ability of RANS equations to produce resolved motions. More specifically, a hybrid model has to respond to the actual resolution with the proper amount of modeled motion to have a balanced total motion–the question is how this can be accomplished. Regarding Q3, a valid question is about how the distribution of LES and RANS modes changes with the grid. One version of asking this is to consider how k_+ depends on $\Delta_+ = \Delta/L_{tot}$. Here, $L_{tot} = L + L_{res}$ is the total (RANS) length scale consisting of modeled and resolved contributions, L and L_{res}, respectively (similar to $k_{tot} = k + k_{res}$). A different version of this question is to ask how the corresponding length scale ratio $L_+ = L/L_{tot}$ depends on Δ_+. This question is equivalent to the question of how the grid affects variables in LES that provide scale information, like L. In particular, there is the question of how L scales with Δ if Δ is not small.

Let us illustrate the specific problem considered for incompressible flow in terms of the popular k–ϵ model for the turbulent kinetic energy k and its dissipation rate ϵ (other turbulence models will be considered below),

$$\frac{Dk}{Dt} = P - \epsilon + D_k, \qquad \frac{D\epsilon}{Dt} = C_{\epsilon_1}\frac{\epsilon^2}{k}\left(\frac{P}{\epsilon} - \alpha^*\right) + D_\epsilon. \tag{1}$$

Here, $D/Dt = \partial/\partial t + U_k\partial/\partial x_k$ refers to the Lagrangian time derivative, which involves the component U_k of the mean velocity. The sum convention is applied throughout this paper, t is time, and x_k is the kth space coordinate. No explicit expression will be used below for the production of turbulent kinetic energy P. A usual parametrization is given by $P = \nu_t S^2$. Here, $S = (2S_{ik}S_{ki})^{1/2}$ is the characteristic shear rate which involves the rate of strain $S_{ik} = (\partial U_i/\partial x_k + \partial U_k/\partial x_i)/2$. The turbulent viscosity is given by $\nu_t = C_\mu k^2/\epsilon$, where C_μ is a constant. In addition, we have $\alpha^* = \alpha$ with $\alpha = C_{\epsilon_2}/C_{\epsilon_1}$. C_{ϵ_1} and C_{ϵ_2} are constants having standard values $C_{\epsilon_1} = 1.44$ and $C_{\epsilon_2} = 1.92$. The turbulent transport terms are given by

$$D_k = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right], \quad D_\epsilon = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_j}\right]. \tag{2}$$

Here, ν refers to the constant molecular viscosity, and σ_k and σ_ϵ are constants.

The inclusion of resolving motions in the RANS equations (1) is possible by considering a variable α^* instead of the constant α. The problem is to determine α^* such that a desired resolution can be realized. The particular challenge is to ensure the functioning of this mechanism for varying resolution levels. Popular solutions to this problem are offered by partially averaged Navier-Stokes (PANS)[6] and partially integrated transport modeling (PITM):[4,5,12] both approaches assume $\alpha^* = 1 + R(\alpha - 1)$, where $R = k_+$.

Mathematically, the appropriate way to address the calculation of α^* is to consider variations of model parameters and implied variations of model variables: the question is which model parameters satisfy variation equations implied by turbulence models. Technical material in this regard is presented in the first four paragraphs of the last six paragraphs of this letter (beginning with *Sections IA-ID*, respectively). The analysis shown in *Section IA* leads to the conclusions that $R = L_+^2$ in $\alpha^* = 1 + R(\alpha - 1)$, which is clearly different from $R = k_+$ applied in PANS-PITM approaches[4–6,12] (see the corresponding discussion below). The analysis in *Section IB* shows that there exists a corresponding detached eddy simulation (DES) k–ϵ model, which is (resolution-wise) equivalent to the PANS-PITM-type k–ϵ model equations (1). The analysis shown in *Section IC* leads to the conclusion that there exists a PANS-PITM-type k–ω model, which is (resolution-wise) equivalent to the PANS-PITM-type k–ϵ model (1). The analysis in *Section ID* shows that there exists a corresponding DES k–ω model, which is (resolution-wise) equivalent to Eq. (1). Within the framework considered [the equations considered and condition of Friess *et al.*[10] that the energy partition variation ($\delta k/k$, $\delta\epsilon/\epsilon$) over the domain is uniform[16]], the results reported in *Sections IA-ID* are exact analytical implications. It is worth noting that the analysis of Friess *et al.*[10] provides the technical framework for obtaining the results shown in *Sections IA-ID*, but the latter results were not reported so far (Friess *et al.*[10] focused on another question, the equivalence of hybrid methods: see the discussion of Q2 above).

Let us use the *Sections IA-ID* results to present observations regarding Q1 considered above. The characterization of the degree of flow resolution is very relevant and rather difficult. Current concepts for the evaluation of the resolution of simulations are based on comparisons of the suitability of parameters that appear to be appropriate.[14] No theory was presented so far providing guidance regarding an optimal resolution measure. A relevant result of the analysis presented here is the mathematical identification of L_+^2 as a resolution measure regarding the models considered. Without involving Eq. (16), L_+^2 was shown to provide information about the grid-induced resolution to the model equations and to control the model's response to the amount of actual resolution. It is of interest that the use of L_+ as a resolution measure agrees with Davidson's analysis[14] of resolution measures, showing that the consideration of velocity correlation functions (the resolution of integral length scales) represents the most promising approach. L_+ (the length scale ratio of modeled LES and RANS motions) may be considered as a Knudsen number: it separates LES and RANS regimes similar to the Knudsen number used in fluid dynamics to separate statistical mechanics and continuum mechanics regimes. With respect to the hybridization of RANS and LES, it needs a reduction of scale variables (length or time scales) to enable resolving simulations. This can be performed in a variety of ways, e.g., by replacing the dissipation time $\tau = k/\epsilon$ in the turbulent viscosity by τk_+.[15] In this regard, the advantage of identifying L_+^2 as the control parameter is the exclusion of empirical solutions to this problem. The use of L_+^2 includes $\epsilon_+ = \epsilon/\epsilon_{tot}$ variations: there is no need to disregard them as often performed in PANS modeling or to include such effects via approximations.[10] A remarkable finding is that L_+^2 was identified as a resolution measure for several (k–ϵ and k–ω) turbulence models and for different (PANS-PITM-type and DES-type) hybridization schemes. In addition to providing support for the suitability of considering L_+^2, this means that there are equivalent formulations of several turbulence models that can produce the same resolution.

Next, let us use the *Sections IA-ID* results to present observations regarding Q2 considered above. The results presented above prove explicitly, first, that many RANS equations can produce resolved motions (i.e., $L_+ \leq 1$) and, second (via the fact that variation equations are satisfied), that modeled and resolved motions can be kept in balance. The essential requirement for both properties is that model coefficients are chosen in consistency with corresponding variation equations. Compared to $\alpha^* = 1 + L_+^2(\alpha - 1)$ used in the k–ϵ model presented here, what are actually the consequences of using $\alpha^* = 1 + k_+(\alpha - 1)$ applied in PANS-PITM methods? The combination of $\alpha^* = 1 + k_+(\alpha - 1)$ with the variation equation (6) implies

$$\frac{\delta k}{k}\Big[\mathrm{N}_\epsilon - \gamma(\mathrm{N}_k - 1 - \mathrm{N}_\epsilon)\Big] = \frac{\delta \epsilon}{\epsilon}\Big[2 + \gamma(1 - \mathrm{N}_{k\epsilon})\Big], \tag{3}$$

where $\gamma = D_k/[(\alpha - 1)k_+\epsilon]$. For independent variations, the only way to satisfy this equation is to neglect ϵ variations and, consequently, to set $\mathrm{N}_\epsilon = 0$. Then, the first term requires $\gamma = 0$, which means that $D_k = 0$. Thus, PANS-PITM methods are (in addition to the neglect of ϵ variations) equivalent to the consideration of homogeneous turbulence. With respect to the k–ω model, we find a corresponding result as the consequence of the PANS-PITM setting $\beta^* = 1 + k_+(\beta - 1)$. As a consequence of PANS-PITM concepts, hybrid models involving $\epsilon_+ \neq 1$ effects and variations of C_μ in the turbulent viscosity need to account for these effects approximately, and they are different depending on the effects considered.[10] On the other hand, there is no need for considering such different models by using the concepts presented here.

Technical material about the grid influence on the resolution, which will be used in the following paragraph to address Q3 considered above, is presented in *Section II* (the paragraph before the last paragraph of this letter). The relationship $L_+ = \Delta_{C+}/(1 + \Delta_{C+}^3)^{1/3}$ between L_+ and Δ_+ considered in *Section II* was already presented before based on spectral arguments.[12,17] However, given its relevance, the purpose of *Section II* is to provide independent (probabilistic) evidence for the validity of $L_+ = \Delta_{C+}/(1+\Delta_{C+}^3)^{1/3}$ for inhomogeneous turbulent flows.

Let us use now the *Section II* results to present observations regarding Q3 considered above. The relationship $L_+ = \Delta_{C+}/(1 + \Delta_{C+}^3)^{1/3}$, denoted as dispersion relation below, represents a simple interpolation between limits. Evidence for its suitability was provided here by taking reference to probabilistic arguments, in particular, the realizability and statistically most-likely (SML) structure of the related probability density function (PDF) $f = dL_+^2/d\Delta_{C+}$. The dispersion relation enables a better insight into the differences between PANS-PITM approaches (using k_+ to reflect the resolution, we have $k_+ = L_+^{2/3}$ if $\epsilon_+ = 1$ is assumed) and the methods described here (using L_+^2). The difference between $L_+^{2/3}$ and L_+^2 is illustrated in Fig. 1. It may be seen that the use of $L_+^{2/3}$ significantly overestimates the amount of modeled motion. To accomplish the same resolution, a much finer grid has to be applied in PANS-PITM methods. This can be expressed by the equality condition $L_+^{2/3}(\Delta_{C+}/F) = L_+^2(\Delta_{C+})$, where F is the grid refinement factor needed in PANS-PITM. By involving the dispersion relation, the evaluation of the equality condition provides $F = [1 + 3\Delta_{C+}^3(1 + \Delta_{C+}^3)]^{1/3}/\Delta_{C+}^2$. By using the dispersion relation, F can be also written $F = \kappa^{1/3}/L_+^2$, where $\kappa = 1 + L_+^3(1 + L_+^3)$. Here, $1 \leq \kappa^{1/3} \leq 3^{1/3} = 1.44$, i.e., F scales first of all with L_+^{-2}. F becomes huge for small L_+, i.e., PANS-PITM approaches produce in the high-cost LES regime huge additional computational cost. Another viewpoint implies the same. A reasonable definition of computational cost reads $n_+ = 1 + N_+$: N_+ is the number of grid points in simulations and $1 + N_+$ normalizes n_+ such that $n_+ = 1$ for RANS. By using Eq. (16), we find $n_+ = L_+^{-3}$. On the other hand, PANS-PITM approaches that need a finer grid (Δ_+/F instead of Δ_+ in $n_+ = 1+\Delta_{C+}^{-3}$) imply $n_+ = L_+^{-9}$. Hence, PANS-PITM methods have a scaling that is six orders of magnitude higher than the scaling of methods presented here.

In summary, this paper reports analytical results regarding the measurement of resolution, equivalence of hybrid methods, and resolution control in hybrid methods. These results are exact within the framework considered [the equations considered and condition of Friess *et al.*[10] that the energy partition variation ($\delta k/k$, $\delta\epsilon/\epsilon$) over the domain is uniform[16]]. In particular, the following answers to the questions Q1-Q3 considered above were obtained. The question of how the amount of resolved (modeled) motion has to be measured is debated over decades.[14] Here, a mathematical explanation was provided showing that the amount of modeled motion is given by L_+. This conclusion was obtained for several turbulence models, meaning also that various turbulence models can produce the same resolution. Strict explanations

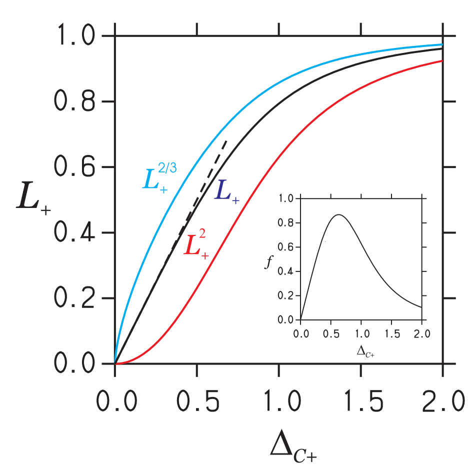

FIG. 1. $L_+^{2/3}$, L_+, and L_+^2 determined by Eq. (16). The dashed line shows Δ_{C+}. The inset shows the PDF $f = dL_+^2/d\Delta_{C+}$.

of why RANS equations can be resolving and how modeled and resolved motions can be kept in balance were not presented so far. Evidence for that was provided here subject to the condition that model coefficients can correctly satisfy variation equations implied by the RANS equations. Various turbulence models can act equivalently if their model coefficients are chosen consistently. Both PANS and PITM approaches are known to produce imbalances in this regard, e.g., regarding the imposed and the actual resolution.[16] The reason for such imbalances was explained by the fact that PANS-PITM concepts apply a relationship between model coefficients and resolution parameters that applies to homogeneous flows and needs the neglect of ϵ variations. So far, there is no agreement about how the grid affects the resolution in hybrid methods and scale variables in LES like L.[12,18–20] The probabilistic interpretation presented here supports the validity of $L_+ = \Delta_{C+}/(1+\Delta_{C+}^3)^{1/3}$. On this basis, it was shown that the use of PITM-PANS concepts has a computational cost scaling with L_+ that is six orders of magnitude higher ($n_+ = L_+^{-9}$) than the cost scaling of methods presented here ($n_+ = L_+^{-3}$), which matters, in particular, in the high-cost LES regime. Regarding the scaling of LES, $L = C_+\Delta/(1+\Delta_{C+}^3)^{1/3}$ explains the scaling of the LES length scale L with Δ.

Section IA. We apply variational analysis to the k–ϵ model equations (1) to find out which α^* is needed to enable a desired resolution. To prepare the following developments, we consider the variations (denoted by δ) of diffusion terms. By following, basically, the analysis of Friess *et al.*,[10] we find $\delta D_k/D_k = \alpha_k^3/\alpha_\epsilon - 1$ and $\delta D_\epsilon/D_\epsilon = \alpha_k^2 - 1$. The relevant condition applied here is that variations of $\alpha_k = 1 + \delta k/k$ and $\alpha_\epsilon = 1 + \delta\epsilon/\epsilon$ in space can be neglected, which is equivalent to looking for model coefficient variations that produce an uniform variation of the energy partition over the domain: see the justification provided by Friess *et al.*[10] and Manceau *et al.*[16] In the first order of approximation, the relations $\delta D_k/D_k = \alpha_k^3/\alpha_\epsilon - 1$ and $\delta D_\epsilon/D_\epsilon = \alpha_k^2 - 1$ read

$$\frac{\delta D_k}{D_k} = N_k\frac{\delta k}{k} - N_{k\epsilon}\frac{\delta\epsilon}{\epsilon}, \qquad \frac{\delta D_\epsilon}{D_\epsilon} = N_\epsilon\frac{\delta k}{k}, \tag{4}$$

where $N_k = 3$, $N_{k\epsilon} = 1$, and $N_\epsilon = 2$. The noticeable difference to the corresponding expressions obtained by Friess *et al.*[10] is the inclusion of ϵ variations. By following the analysis of Friess *et al.*,[10] we neglect (with respect to the variational analysis performed here) the left-hand side (LHS) of both equations in the k–ϵ model (1), which is equivalent to assuming that both k and ϵ are in equilibrium along mean streamlines. By using $P = \epsilon - D_k$ in the ϵ equation, we obtain

$$0 = C_{\epsilon_1}\frac{\epsilon^2}{k}\left(1 - \frac{D_k}{\epsilon} - \alpha^*\right) + D_\epsilon. \tag{5}$$

The function α^* can be found by considering the variation of Eq. (5),

$$\delta\alpha^* = (\alpha^* - 1)\left[(1+N_\epsilon)\frac{\delta k}{k} - 2\frac{\delta\epsilon}{\epsilon}\right] - T_\alpha, \tag{6}$$

where $T_\alpha = \epsilon^{-1}D_k[(1 - N_{k\epsilon})\delta\epsilon/\epsilon + (N_k - 1 - N_\epsilon)\delta k/k]$. Here, Eq. (5) was used to replace D_ϵ. The use of $N_k = 3$, $N_{k\epsilon} = 1$, and $N_\epsilon = 2$ finally implies

$$\frac{\delta\alpha^*}{\alpha^* - 1} = 3\frac{\delta k}{k} - 2\frac{\delta\epsilon}{\epsilon} = 2\frac{\delta L}{L}, \tag{7}$$

where $L = k^{3/2}/\epsilon$ is the turbulence model length scale. The integration of Eq. (7) from the RANS state to a state with a certain level of resolved motion results in $\int_\alpha^{\alpha^*} dx/(x-1) = 2\int_{L_{tot}}^{L} dy/y$. The evaluation of the latter expression implies

$$\alpha^* = 1 + R(\alpha - 1), \tag{8}$$

where $R = L_+^2$.

Section IB. It is of interest to look at other versions of setting up hybrid models within the k–ϵ model framework. According to the DES concept,[10,12] we may consider

$$\frac{Dk}{Dt} = P - \psi_\alpha\epsilon + D_k, \qquad \frac{D\epsilon}{Dt} = C_{\epsilon_1}\frac{\epsilon^2}{k}\left(\frac{P}{\epsilon} - \alpha\right) + D_\epsilon. \tag{9}$$

Variations of resolved and modeled motions can be included here via the function ψ_α, which has to be determined. In correspondence to the analysis presented before, we neglect the LHS of both equations and use $P = \psi_\alpha\epsilon - D_k$ in the ϵ equation so that $0 = C_{\epsilon_1}k^{-1}\epsilon^2(\psi_\alpha - D_k/\epsilon - \alpha) + D_\epsilon$. The comparison with Eq. (5) reveals the equivalence of models, provided we have

$$\alpha^* = 1 + \alpha - \psi_\alpha. \tag{10}$$

Section IC. It is also of interest to look at corresponding implications for other two-equation turbulence models, like the k–ω model,

$$\frac{Dk}{Dt} = P - \epsilon + D_k, \qquad \frac{D\omega}{Dt} = C_{\omega_1}\omega^2\left(\frac{P}{\epsilon} - \beta^*\right) + D_\omega, \tag{11}$$

which involves $\omega = \epsilon/k$. Here, we have $P = \nu_t S^2$ with $\nu_t = C_\mu k/\omega$ and $\beta^* = \beta$ with $\beta = C_{\omega_2}/(C_k C_{\omega_1})$. The diffusion term in the ω equation reads $D_\omega = \partial[(\nu + \nu_t/\sigma_\omega)\partial\omega/\partial x_j]/\partial x_j$. The constants involved have the values $C_{\omega 1} = 0.49$, $C_{\omega 2} = 0.072$, $C_k = 0.09$, $C_\omega = 1.1$, $\sigma_\omega = 1.8$. Usually, a cross diffusion term $D_{\omega c} = C_\omega k^{-1}(\nu + \nu_t)[\partial k/\partial x_j][\partial\omega/\partial x_j]$ is added to Eq. (11). However, it does only affect the model behavior near boundaries (it acts like a damping function). Thus, it will be neglected for the following analysis. The hybridization of the RANS equations (11) requires to determine a variable β^* to enable a certain resolution. As performed with respect to the k–ϵ models, we neglect both LHS's in Eqs. (11) and use $P = \epsilon - D_k$ in the ω equation to obtain $0 = C_{\omega_1}\omega^2(1 - D_k/\epsilon - \beta^*) + D_\omega$. The variational analysis of this equation can be performed by following the corresponding analysis of k–ϵ models. We find $\delta D_\omega/D_\omega = \alpha_k - 1$. In the first order of approximation, we obtain $\delta D_\omega/D_\omega = N_\omega\delta k/k$, where $N_\omega = 1$. On this basis, we find

$$\delta\beta^* = (\beta^* - 1)\left[(2+N_\omega)\frac{\delta k}{k} - 2\frac{\delta\epsilon}{\epsilon}\right] - T_\beta, \tag{12}$$

where $T_\beta = \epsilon^{-1}D_k[(1 - N_{k\epsilon})\delta\epsilon/\epsilon + (N_k - 2 - N_\omega)\delta k/k]$. Combined with $N_k = 3$, $N_{k\epsilon} = 1$, $N_\omega = 1$, and $R = L_+^2$, the latter relation implies

$$\beta^* = 1 + R(\beta - 1). \tag{13}$$

Section ID. In correspondence to the DES version of the $k-\epsilon$ model, we may consider

$$\frac{Dk}{Dt} = P - \psi_\beta \epsilon + D_k, \qquad \frac{D\omega}{Dt} = C_{\omega_1}\omega^2\left(\frac{P}{\epsilon} - \beta\right) + D_\omega, \tag{14}$$

where ψ_β needs to be determined. We neglect both LHS's and use $P = \psi_\beta\epsilon - D_k$ in the ω equation to obtain $0 = C_{\omega_1}\omega^2(\psi_\beta - D_k/\epsilon - \beta) + D_\omega$. The comparison with the corresponding $k-\omega$ equation $0 = C_{\omega_1}\omega^2(1 - D_k/\epsilon - \beta^*) + D_\omega$ reveals the consistency of models, provided we have

$$\beta^* = 1 + \beta - \psi_\beta. \tag{15}$$

Section II. Let us consider the calculation of L_+, which is needed in the equations presented above. L_+ can be prescribed or directly calculated during the simulation. There are versions of performing this (using relations of L_+ to k_+, $\epsilon_+ = \epsilon/\epsilon_{tot}$, and M_+, where M_+ refers to the modeled-to-total ratio of the Reynolds stress considered in a coordinate-invariant manner).[20] For example, L_+ can be determined via its relationship to M_+ combined with a prescribed value of M_+.[20] Nevertheless, there is the question of how L_+ is related to the grid, which is relevant because of both practical and theoretical reasons. There are questions regarding the setup of simulations, e.g., what approximately is k_+ for a certain grid and what is the change of the energy ratio if other grids are used. More importantly, the question of how L_+ is related to Δ_+ is relevant to the theoretical setup of LES and hybrid methods. For LES, there is the problem to explain the scaling of the model length scale L with Δ, see Refs. 15 and 20. With respect to hybrid RANS-LES, there is the question of how the mode distribution changes with the grid. A significant number of relationships between L_+ and Δ_+ are in use, see the reviews in Refs. 12, 18, and 19. Only a few suggestions for solving this problem were presented as being supported by theory.[4,5,17] Written in terms of L_+, the recent version of this development, which involves any number p, reads[17]

$$L_+ = \Delta_{C+}/\left(1 + \Delta_{C+}^p\right)^{1/p}, \tag{16}$$

where $\Delta_{C+} = C_+\Delta_+$. In practice, $p = 3$ is applied.[12] The constant C_+ was determined to be $C_+ = (3C_K/2)^{3/2}/\pi$, where C_k refers to the Kolmogorov constant. Values $C_k = (1.3, 1.43, 1.5)$, e.g., imply $C_+ = (0.876, 1.0, 1.074)$. For sufficiently small Δ_+, Eq. (16) implies $L_+ = \Delta_{C+}$ (see Fig. 1 below). The relation $L_+ = \Delta_{C+}$ is simply a consequence of calculating L_+ by integration over the Kolmogorov spectrum.[5] Based on empirical reasoning, the same expression was suggested by Girimaji and Abdol-Hamid[21] with the noticeable difference that $C_+ = C_\mu^{-3/4} = 6.09$. An argument in favor of $C_+ \approx 1$ is that this value agrees in the LES limit very well with usual parametrizations of the dissipation rate in LES equations: we have $\epsilon = k^{3/2}/L = k^{3/2}/[C_+\Delta]$. A rewriting of Eq. (16) reads $L_+^{-p} = 1 + \Delta_{C+}^{-p}$, which is simply an interpolation between the limits of small and large Δ_+ values (Δ_{C+} and unity). Proof of the validity of Eq. (16) requires support for the interpolation structure of this equation. The latter can be obtained in the following way. L_+^2, which controls the RANS-LES transition, represents a cumulative distribution function (CDF) changing from zero to unity. From a probability perspective, this CDF should be realizable, i.e., implied by an existing PDF. In particular, this PDF should have existing moments, at least a mean and variance (the existence of finite moments of such a PDF is not ensured in general: for example, the PDF implied by an earlier L_+^2 model[4] does not have existing moments). The PDF considered is given by $f(t) = dL_+^2/d\Delta_{C+} = 2t^{1/p}(1 - t)^{1+1/p}$ (see the inset in Fig. 1). Here, $0 \le t \le 1$ is defined via $1 - t = (1 + \Delta_{C+}^p)^{-1}$. For $p > 1$, the PDF $f(t[\Delta_{C+}])$ has an existing mean, $m_f = 2p^{-1}\int_0^1[t/(1-t)]^{1/p}t^{2/p-1}dt$, and for $p > 2$, it has an existing variance, $\sigma_f^2 = 2p^{-1}\int_0^1[t/(1-t)]^{2/p}t^{2/p-1}dt - m_f^2$. The evaluation of m_f and σ_f^2 results in $m_f = 2p^{-1}B(3/p, 1 - 1/p)$ and $\sigma_f^2 = 2p^{-1}B(4/p, 1 - 2/p) - m_f^2$, respectively. $B(A, B) = \int_0^1 t^{A-1}(1-t)^{B-1}dt$ is the beta function: see Ref. 22. A further analysis reveals that this PDF has theoretical support; it represents a SML PDF that maximizes the corresponding entropy on the probability space considered.[23] The setting $p = 3$ (which implies a PDF mean $m_f = 1$) can be justified as follows. According to Eq. (16), the modeled volume ratio of turbulence elements reads $L_+^3 = 1/(1 + N_+^{p/3})^{3/p}$, where $N_+ = \Delta_{C+}^{-3}$ characterizes the number of grid points in three-dimensional simulations ($\Delta_+ = \Delta/L_{tot}$). This implies $dL_+^3/dN_+ = -N_+^{p/3-1}L_+^{3+p}$. To exclude a divergence of dL_+^3/dN_+, e.g., for large N_+, we have to require that dL_+^3/dN_+ is independent of N_+. The latter is the case for $p = 3$.

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