Homework 3 (MATH 5490-01) **Name (Print):**
Due date: Monday, Oct. 19, 2009

Consider the following equations for x_n and y_n:

$$x_n = x_{n-1} + \Delta t\, y_{n-1}, \qquad (1)$$

$$y_n = a\, y_{n-1} + b + r\, \varepsilon_{n-1}. \qquad (2)$$

Here, ε_k is normally distributed with the properties

$$\langle \varepsilon_k \rangle = 0, \qquad \langle \varepsilon_k \varepsilon_m \rangle = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}. \qquad (3)$$

The other variables a, b, r, and Δt are any deterministic model parameters.

1. Derive the solution (x_n, y_n) such that x_n and y_n are explicit functions of n, x_0, y_0, a, b, r, Δt and ε_0, ε_1, ε_2, ….

2. Use the solutions for x_n and y_n to show that

$$\langle x_n \rangle = \langle x_0 \rangle + \Delta t \langle y_0 \rangle \frac{1-a^n}{1-a} + \frac{\Delta t\, b}{1-a}\left[n - \frac{1-a^n}{1-a} \right], \qquad (4)$$

$$\langle y_n \rangle = a^n \langle y_0 \rangle + b \frac{1-a^n}{1-a}, \qquad (5)$$

3. Use the solutions for x_n and y_n to show that

$$\langle \tilde{x}_n^2 \rangle = \langle \tilde{x}_0^2 \rangle + (\Delta t)^2 \langle \tilde{y}_0^2 \rangle \frac{(1-a^n)^2}{(1-a)^2} + \frac{(\Delta t)^2 r^2}{(1-a)^2}\left[n - 2\frac{1-a^n}{1-a} + \frac{1-a^{2n}}{1-a^2} \right], \qquad (6)$$

$$\langle \tilde{x}_n \tilde{y}_n \rangle = \Delta t \langle \tilde{y}_0^2 \rangle \frac{1-a^n}{1-a} a^n + \frac{\Delta t\, r^2}{1-a}\left[\frac{1-a^n}{1-a} - \frac{1-a^{2n}}{1-a^2} \right], \qquad (7)$$

$$\langle \tilde{y}_n^2 \rangle = a^{2n} \langle \tilde{y}_0^2 \rangle + r^2 \frac{1-a^{2n}}{1-a^2}. \qquad (8)$$

4. Plot the means and variances (4) – (8) as functions of n where a = 0.5, b = 0, r = 0.3, and Δt = 0.01.

5. Explain how the relations (4) – (8) can be used to derive model parameters from given statistics.